Erzeugung und Eigenschaften von kurzen und ultra-kurzen Lichtpulsen

INAUGURAL-DISSERTATION

der philosophisch-naturwissenschaftlichen Fakultät der Universität Bern
zur Erlangung der Doktorwürde

vorgelegt von

ANDREAS ALFRED GRÜTTER
Seeberg (Be)

Von der philosophisch-naturwissenschaftlichen Fakultät
auf Antrag von Herrn Prof. Dr. K. P. Meyer angenommen.

Bern, den 2. Juli 1970 Der Dekan: Prof. Dr. J. Geiss

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The Physical Review, Vol. 185, 629 (69)
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bestehend aus den folgenden Einzelarbeiten:

I. Teil:

«Imperfectly Mode-Locked Laser Emission and Its Effects on Nonlinear Optics»
by A. A. Grütter, H. P. Weber and R. Dändliker

The Physical Review, Vol. 185, No. 2, 629—643, 10 September 1969

II. Teil:

«Statistical Amplitude and Phase Variations in Mode-Locked Lasers»
by R. Dändliker, A. A. Grütter and H. P. Weber

Journal of Quantum Electronics, Vol. QE-6, 687—693, No. 11, November 1970

III. Teil:

«The Influence of Spatial Hole-Burning in Single Mode Q-Switch Operation»
by H. Weber and A. A. Grütter

Opto-Electronics, Vol. 2, No. 2, 63—67, May 1970

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Lebenslauf

Geboren am 1. Mai 1942, durchlief ich die regulären Schulen von der Primarschule bis zum Gymnasium, das ich 1962 mit der Matura Typus C abschloß. Zwischendurch (1959/1960) ermöglichte mir ein Stipendium einen einjährigen Aufenthalt an einer amerikanischen High School im Rahmen des AFS-Studenten-Austauschprogramms. Das Studium der Physik, Mathematik und Astronomie begann ich anschließend an die Gymnasialzeit an der Universität Bern, wo ich 1967 mit einer Arbeit über Symmetrieoperationen der Elementarteilchenphysik am Institut für theoretische Physik das Lizentiat erwarb. In der nachfolgenden Zeit als Assistent am Institut für angewandte Physik hatte ich Gelegenheit, mich mit Fragen der Lichterzeugung und Lichtstatistik zu befassen, wie sie insbesondere seit der Entwicklung des Lasers aktuell geworden sind. Die vorliegende Dissertation umfaßt einzelne Publikationen, die dabei unter der Leitung und in Zusammenarbeit mit den Herren Dr. H. Weber, Dr. R. Dändliker und Dr. H. P. Weber entstanden sind.

Andreas A. Grütter

Liste der Publikationen

1. H. P. Weber, R. Dändliker und A. A. Grütter, «Das Fluktuationsverhalten von Signalen, die durch optisches Mischen entstanden sind», Z. Angew. Math. Phys. *20,* 571—572, 1969.

2. R. Dändliker, H. P. Weber and A. A. Grütter, «Influence of Systematic Phase Deviations on the Output of Mode-Locked Lasers», Z. Angew. Math. Phys. *20,* 572—574, 1969.

3. A. A. Grütter, R. Dändliker and H. P. Weber, «The Output Intensity of a Non-Ideally Mode-Locked Laser», Z. Angew. Math. Phys. *20,* 574—577, 1969.

4. A. A. Grütter, H. P. Weber and R. Dändliker, «Imperfectly Mode-Locked Laser Emission and Its Effects on Nonlinear Optics», Phys. Rev. *185,* 629—643, 1969. (Bestandteil der vorgelegten Dissertation.)

5. R. Dändliker, A. A. Grütter and H. P. Weber, «Statistical Amplitude and Phase Variations in Mode-Locked Lasers», IEEE J. Quantum Electron. QE-6, 687—693, 1970. (Bestandteil der vorgelegten Dissertation.)

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7. A. A. Grütter and H. Weber, «Pulse Shaping Effects in Q-Switched Lasers», appears in Opto-Electronics 1971.

Reprinted from THE PHYSICAL REVIEW, Vol. 185, No. 2, 629–643, 10 September 1969
Printed in U. S. A.

Imperfectly Mode-Locked Laser Emission and Its Effects on Nonlinear Optics

A. A. GRÜTTER, H. P. WEBER,* AND R. DÄNDLIKER†
Institute of Applied Physics, University of Berne, Berne, Switzerland
(Received 10 March 1969)

The output of an ideally mode-locked laser, namely, one having equal phase angles of the modes, consists of a train of bandwidth-limited short pulses. The influence of deviations from equal phase angles on the laser output is investigated. Random distributions of the phases of the modes around equal phase angles introduce only limited fluctuating background, whereas the duration of the pulses remains bandwidth-limited. A systematic deviation of the phase angles at least quadratic with mode number is necessary for lengthening of the pulses. The results obtained by the superposition of these two effects are consistent with the experiments published so far. The efficiency of second- and higher-harmonic generation is discussed by help of the moments of the intensity probability distributions. It is shown how these distributions change as a result of harmonic generation and optical mixing. From intensity-correlation measurements, the moments of the intensity probability distribution can be obtained directly, as well as information on the time behavior of the light field. Experimental arrangements used to measure intensity correlations are described.

I. INTRODUCTION

A LASER light source is believed to emit a coherent light beam of intensity varying slowly in time, but only if the laser oscillates in a single mode. However, if the laser oscillates in several axial modes (off-axial modes are not considered, since their addition only introduces geometrical problems), the light intensity is modulated and becomes, with an increasing number of independently oscillating modes, more and more irregularly fluctuating. It can be shown that in the limit of an infinite number of independent modes, the emission becomes completely "chaotic," like a Gaussian light field, as predicted by Mandel and Wolf,[1] and has the same fluctuations as the light of a thermal source transmitted through a suitable narrow-band optical filter.

The fluctuations in the laser intensity are of great importance for nonlinear interactions of light,[2–5] since the efficiency of such interactions depends on the instantaneous intensities of the light fields. The influence of statistical fluctuations of the intensity on optical second- and higher-harmonic production,[2,3] as well as on optical mixing, arising from multimode oscillation, is discussed. It will be demonstrated that the average efficiency of the harmonic generation of lasers with several modes oscillating simultaneously depends predominantly on the kind of intensity fluctuations. For these reasons, the intensity fluctuations of multimode lasers and of higher harmonics generated by multimode lasers will be discussed.

Recently, several authors investigated the time

* Present address: Bell Telephone Laboratories, Inc., Holmdel, N. J.
† Present address: Philips Research Laboratories, N. V. Philips' Gloeilampenfabrieken, Eindhoven, The Netherlands.

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behavior of the output of multimode lasers.[6–17] For the investigation of the time duration of fast fluctuations (some 10^{-12} sec), only methods based on the measurement of autocorrelations of the time-dependent intensity can be employed, since the known electronic detection devices are not fast enough. Such methods were suggested and employed for the evaluation of the time duration of ultrashort light pulses. Some authors use second-harmonic generation and, in fact, measure the autocorrelation function of the light intensity,[6–8,15,18] whereas others use two-photon absorption[9–14,16,17] and measure linear combinations thereof or higher-order correlation functions and corresponding linear combinations.

The theoretical background of the measurement methods has been treated in great detail.[19–23] The interpretation of the experimental results shows that a variety of output characteristics was found. From experiments using saturable dyes for mode locking and second-harmonic generation for the measurement, the existence of isolated single short pulses is affirmed.[6,15] Results obtained with two-photon fluorescence corresponding to nonisolated short pulses and also to isolated pulses have been reported. Similar results, corresponding to large fluctuations, as in nonideal mode locking, were also reported from investigations of Q-switched lasers using rotating mirrors[11] and even from experiments with long-pulse (non-Q-switched) lasers.[12] On the other hand, for Q-switched lasers with a rotating mirror, intensity fluctuations corresponding to Gaussian light were reported.[14]

This paper aims to investigate how the output of a laser behaves for different degrees of mode locking[23,24] and how it can be characterized by measurements. Section II gives examples of computed time-dependent-output intensities $I(t)$ of a multimode laser for different degrees of mode locking. A model consistent with the different measurements is described. In Sec. III, corresponding intensity probability distributions are discussed. These distributions characterize the statistical behavior of the light emissions. They contain all the information necessary to calculate the relative average efficiency of optically nonlinear effects for different types of light emission. These distributions are found to be completely different for the various laser outputs considered. Further, in Sec. IV, the relation between the statistical properties of the fundamental light field and the generated higher harmonics is analyzed; in particular, it is shown that the second harmonic of Gaussian light is no longer Gaussian. Unfortunately, there are as yet no experiments known to measure such probability distributions directly. Only their lowest-order moments can be determined experimentally, e.g., by comparing the average intensities of second and higher harmonics with the corresponding intensities of a light field having known fluctuations and the same fundamental intensity and geometry,[25,26] or by measuring photon-counting distributions[27] if the electronic resolution time is shorter than the fluctuation time of the light field.

Another possible method of describing the fluctuations of a light field by measuring its intensity-correlation functions is discussed in Sec. V. The corresponding experimental techniques are the same as those mentioned above, and they are used for the measurement of the time duration of ultrashort light pulses. It will be shown that from the measured intensity-correlation function of a given order, the moment of the intensity probability distribution of the same order can be obtained. In addition, contained in the correlation function is some further information of the time behavior of a light field which is not contained in the probability distribution. In Sec. VI, possible experimental arrangements[6,7,9,18,28] used to measure intensity-correlation functions are described and their performance is discussed.

II. TIME DEPENDENCE OF THE INTENSITY

The multimode emission of a laser light source may be described by the analytic signal $V(t)$ in the form

$$V(t)=\sum_{n=-N}^{+N} a_n(t)\exp[i\omega_n t+\varphi_n(t)], \qquad (1)$$

where $a_n(t)$, ω_n, and $\varphi_n(t)$ are the real amplitude, the frequency, and the phase of the nth mode. The amplitudes and the phase angles are slowly varying in time and are considered in the following discussion to be constant during the emission. The frequencies ω_n are

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considered to be separated by a constant amount Ω, and can therefore be written as $\omega_n=\omega_0+n\Omega$, where ω_0 is the center frequency. These simplifications do not restrict the generality of the following discussions. The instantaneous intensity $I(t)$ of the light source becomes

$$I(t)=V(t)V^*(t)$$
$$=\sum_{n,m} a_n a_m \exp\{i[(n-m)\Omega t+\varphi_n-\varphi_m]\}. \quad (2)$$

By use of this relation, the light intensities can be calculated if the amplitudes a_n and the phase angles φ_n are given.

At this point it is necessary to comment on the different time- and ensemble-averaging processes. Two characteristic times of the light emission have to be distinguished: first, the repetition time T of the intensity $I(t)$ equal to the period of the lowest beat frequency, which is the mode separation ($T=2\pi/\Omega$); second, the coherence time T_{coh} of the light field, which is equal to the inverse of the oscillating bandwidth ($T_{\text{coh}}=\Delta\nu^{-1}$). Since the oscillating bandwidth $\Delta\nu$ determines the highest beat frequency, T_{coh} is the characteristic time of the most rapid variations of the intensity $I(t)$. Thus, measurements of the instantaneous intensity have to be taken within a shorter time interval than the coherence time T_{coh}. For fixed amplitudes a_n and phases φ_n, a time-independent mean value of the intensity or any function of the intensity can be obtained by averaging in time over an interval equal to the repetition time T. This time average will be denoted by indexed angle brackets ($\langle\ \rangle_t$). If statistical distributions of the phases φ_n and/or the amplitudes a_n are considered, the time behavior of $I(t)$ and functions of $I(t)$ are, in general, different for any specific sample of the phases and amplitudes. Averaging over all samples of this distribution yields the so-called ensemble average, which, in general, is still time-dependent. The ensemble average will be denoted by angle brackets without index ($\langle\ \rangle$). In calculations of the time- and ensemble-averaged values, the results are independent of the order in which the two averaging processes are carried out.

The intensity, as a function of time, was numerically calculated for 31 modes with amplitudes given by a Gaussian envelope of a standard deviation corresponding to 8 modes and with different distributions of the phase angles φ_n. Since the mean intensity $\langle I\rangle_t$ is independent of the actual phases, the equality $\langle\langle I\rangle\rangle_t=\langle I\rangle_t$ holds, where $\langle\langle I\rangle\rangle_t$ is the time and ensemble average of the intensity. The results are shown in Figs. 1–3 and will be discussed in the following.

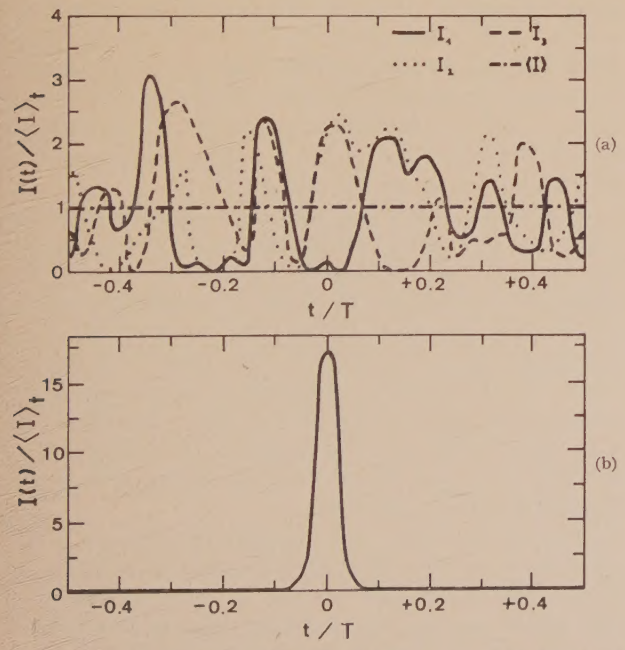

FIG. 1. Instantaneous intensity of a laser with 31 modes of amplitudes, having a Gaussian envelope, normalized to the mean intensity $\langle I\rangle_t$. T is the repetition time of the intensity. (a) Three examples (I_1–I_3) with phase angles randomly distributed in the interval $-\pi<\varphi<+\pi$ and their ensemble-averaged intensity $\langle I\rangle$. (b) Intensity of the same laser modes ideally mode-locked.

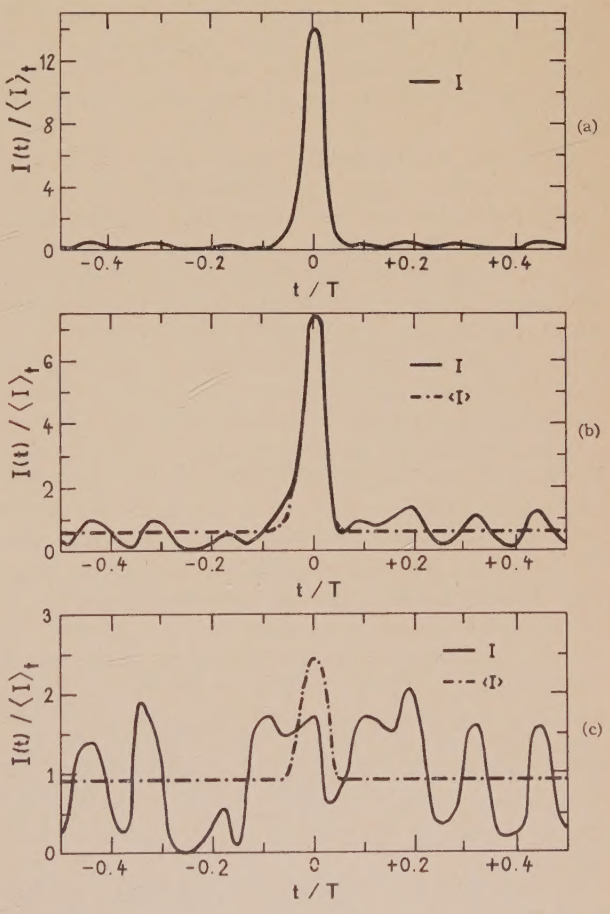

FIG. 2. Example of the instantaneous intensity of a laser with 31 modes of amplitudes having a Gaussian envelope and phase angles φ randomly distributed in different intervals: (a) $-\frac{1}{4}\pi<\varphi<+\frac{1}{4}\pi$, (b) $-\frac{1}{2}\pi<\varphi<+\frac{1}{2}\pi$, (c) $-\frac{3}{4}\pi<\varphi<+\frac{3}{4}\pi$. T is the repetition time, $\langle I\rangle_t$ the mean intensity, and $\langle I\rangle$ the ensemble-averaged ntensity. Note that the scale of $I(t)$ differs in the three figures.

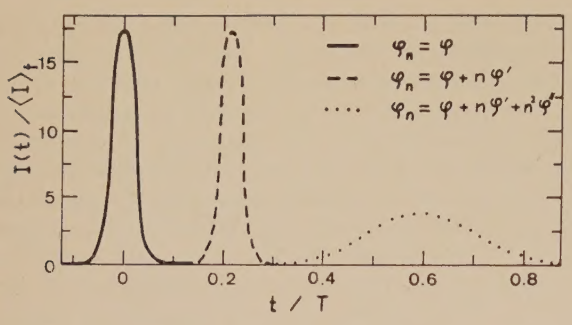

FIG. 3. Instantaneous intensity of a laser with fixed-phase angles and a systematic deviation from ideal mode-locking.

First, the phase angles φ_n are chosen with random distribution in the interval $-\pi < \varphi_n \leq \pi$. As seen from Fig. 1(a) the fluctuations are very pronounced and completely random. The emission becomes periodic in time because of the constant-difference frequency Ω of the modes. The ensemble average of the intensity $\langle I(t)\rangle$ over the phase distribution has a time-independent constant value which is equal to the time average of the intensity $\langle I(t)\rangle_t$. In contrast to this example, Fig. 1(b) shows the result for equal phase angles $\varphi_n = \varphi$ of all modes. This emission consists of a train of equally separated short pulses, corresponding to complete mode locking.

In several experiments, isolated short pulses were observed in the emission of multimode lasers, but the comparison with the theoretical limit of ideal mode locking sometimes shows unexpected significant deviations. Therefore, several theoretical possibilities for deviation from ideal mode locking will be discussed and compared with the known experimental results.

In the emission of a multimode laser, a small part of the spectrum may be locked to a short pulse while the rest of the spectrum is free running,[23] or the spectrum can be divided into small parts, each of them locked to short pulses independently. Beside the possibility that the emission consists of a superposition of the extreme cases with completely undermined and completely determined phase angles, an alternative situation of limited phase variations can be considered with phase angles φ_n randomly distributed in a restricted interval smaller than 2π. The results for the intervals $(-\frac{1}{4}\pi, +\frac{1}{4}\pi)$, $(-\frac{1}{2}\pi, +\frac{1}{2}\pi)$, and $(-\frac{3}{4}\pi, +\frac{3}{4}\pi)$ are given in Fig. 2 as solid lines. A comparison with the ideally mode-locked case in Fig. 1(b) shows that the duration of the pulses is nearly unchanged and that only the fluctuations between the pulses have increased intensity due to increasing the interval of the statistical phase variations.

In the following, it is shown that the time-dependent ensemble average of the intensity $\langle I(t)\rangle$ can be expressed as a linear combination of the ensemble-averaged intensities in the two limiting cases of completely undetermined and completely determined phase angles. The ensemble average of the time- and phase-dependent intensity $I(t,\varphi_n,\varphi_m)$ over randomly distributed phase angles φ_n and φ_m in the interval $-\phi < \varphi_n, \varphi_m < +\phi$ is given by

$$\langle I(t)\rangle = \frac{1}{4\phi^2}\int_{-\phi}^{\phi} d\varphi_n \int_{-\phi}^{+\phi} d\varphi_m I(t,\varphi_n,\varphi_m). \tag{3}$$

By use of Eq. (2) for $I(t,\varphi_n,\varphi_m)$, this can be written as

$$\langle I(t)\rangle = \sum_{n=-N}^{+N} a_n^2 + \frac{1}{4\phi^2}\sum_{n\neq m} a_n a_m e^{i(n-m)\Omega t} \times \int_{-\phi}^{+\phi} e^{i\varphi_n} d\varphi_n \int_{-\phi}^{+\phi} e^{-i\varphi_m} d\varphi_m. \tag{4}$$

Since all phases φ_n are randomly distributed in the same interval, the values of the integrals are independent of m and n and Eq. (4) becomes

$$\langle I(t)\rangle = \sum_n a_n^2 + \frac{\sin^2\phi}{\phi^2}\sum_{n\neq m} a_n a_m e^{i(n-m)\Omega t}. \tag{5}$$

Since the ensemble-averaged intensity $\langle I\rangle$ for $\phi = \pi$ is equal to the time-averaged intensity $\langle I\rangle_t$ and the ensemble-averaged intensity for $\phi = 0$ is equal to the time-dependent intensity $I_p(t)$ of the ideally mode-locked pulse with the assumed amplitudes a_n, Eq. (5) becomes

$$\langle I(t)\rangle = \langle I\rangle_t + (\sin^2\phi/\phi^2)[I_p(t) - \langle I\rangle_t]. \tag{6}$$

This result shows that the variation of the intensity is always of the same shape as the ideally mode-locked pulse and that the fluctuations in the phase angles only introduce a background intensity of the type of random-phase emission. The dashed curves in Fig. 2 represent the ensemble average of the intensity calculated with Eq. (6) for $\phi = \frac{1}{4}\pi$, $\phi = \frac{1}{2}\pi$, and $\phi = \frac{3}{4}\pi$.

In all these cases, only randomly distributed phase angles φ_n in a given interval were considered. In Appendix A, it is shown that if the phases are statistically distributed according to a more complicated phase-distribution function, then the ensemble-averaged intensity $\langle I(t)\rangle$ is still given by superposition of the two cases: (1) the intensity according to completely determined phase angles, and (2) the mean intensity $\langle I\rangle_t$, analogous to Eq. (6). The weighting factor is given by a special value of the Fourier transformation of the phase distribution function.

It is further shown that this superposition law is still valid under an even more general assumption. Consider a light field determined uniquely by given values φ_k^0 of the phase angles and yielding the time-dependent intensity $I_{\text{det}}(t)$. If statistical fluctuations of the phases around φ_k^0 for each mode are introduced according to a common phase-distribution function, the ensemble-averaged intensity becomes a superposition of $I_{\text{det}}(t)$ and the mean intensity $\langle I\rangle_t$. Thus, in this very general

case of phase distributions, the shape of the ensemble-averaged modulation of the emitted intensity is the same as the shape of $I_{\mathrm{det}}(t)$.

It is therefore of interest to investigate the shapes $I_{\mathrm{det}}(t)$ of such determined distributions of the phase angles. In particular, the effect of a systematic deviation from the ideal situation of equal phases of all modes ($\varphi_n=\varphi$) shall be studied. This deviation is assumed to be expanded in a power series with respect to the mode separation n from the center frequency ω_0,

$$\varphi_n=\varphi+n\varphi'+n^2\varphi''+\cdots. \tag{7}$$

The numerically computed results for the three cases $\varphi'=\varphi''=0$; $\varphi'=-1.356$, $\varphi''=0$; and $\varphi'=-0.0576$, $\varphi''=0.08$ are shown in Fig. 3. A linear deviation with mode number obviously results only in a shift of the ideally mode-locked pulse in time by $\tau=-\varphi'/\Omega$, whereas the quadratic term introduces a change of pulse shape.

As shown in Appendix B, linear deviation of the phases is introduced if an ideally mode-locked pulse travels through a dispersive medium with frequency-independent group velocity or with only linear dispersion $d\eta/d\lambda$ of the refractive index η with respect to the vacuum wavelength λ. To change the pulse shape and to increase the pulse width, a deviation at least quadratic with the mode number is necessary. A change of the width of a pulse propagating through a dispersive element is caused by the frequency dependence of the group velocity, which means in optical terms, by the quadratic ($d^2\eta/d\lambda^2$) or higher-order dispersion of the refractive index η with respect to the vacuum wavelength λ.

Recently, Treacy[29,30] was able to "compress" the pulses of a not ideally mode-locked Nd-glass laser to widths close to the theoretical limit of the inverse bandwidth by passing the output beam through a pair of parallel diffraction gratings of appropriate separation. Since this arrangement introduces a systematic change of the phases of the modes with an essential quadratic dependence on the mode number, this experiment proves that the deviation from complete mode locking was of the same systematic kind as discussed above. In theoretical considerations of the mechanism of generation of mode-locked laser pulses, Cubeddu and Svelto[31] have found independently this same output characteristic originating from the dispersion of the host laser material. They also showed that this quadratic type of phase deviation corresponds to a linear sweep of the center frequency, a so-called chirp, which was used by Giordmaine *et al.*[32] to compress bandwidth-limited optical pulses.

As mentioned before, in addition to such a quadratic type of phase deviation, large fluctuations around these fixed phase angles may exist still, introducing only some fluctuating background and letting the shape of the pulse remain essentially unchanged. Such an output would be consistent with both the experimental results[9,11,12] pointing to a background as well as with the experiment of Treacy[29,30] showing phase modulation.

III. INTENSITY PROBABILITY DISTRIBUTIONS AND THEIR MOMENTS

If the time dependence of the intensity of a light field is known, the efficiency of nonlinear interactions of this field with matter can be calculated. Examples are optical second- and third-harmonic generation, the gain of a Raman amplifier, or the field strength in self-focused filaments. Unfortunately, only in rare cases is the time dependence of the intensity known, namely, if there are only one or two modes oscillating or if ideal mode locking is assumed. For other cases, usually the spectral width of the emission and thus the frequencies of the fluctuation oscillations are too fast to be controlled by electronic means. In order to control the effects arising from the fluctuations in nonlinear optical experiments, Ducuing and Bloembergen[2] introduced the experimental technique of the harmonic monitor. Its output can be used as a reference to normalize the measurements independently of the effects arising from the unknown fluctuations.

Since, in general, the time dependence of the intensity is not explicitly known, it is useful to characterize the light intensities by a probability distribution $p(I)$, such that the probability of finding an intensity I between I and $I+dI$ is $p(I)dI$, if I is measured during times short in comparison to the fluctuation time or coherence time, i.e., the inverse of the total oscillating bandwidth.[33] The intensity measurements are taken from different samples of the ensemble and at different times, separated by more than the coherence time. It should be noted that

$$I\geq 0;\quad p(I)\geq 0,\ \text{all } I;$$
$$\int_{-\infty}^{+\infty} p(I)dI=1;\quad \int_{-\infty}^{+\infty} Ip(I)dI=\langle\langle I\rangle\rangle_t, \tag{8}$$

where $\langle\langle I\rangle\rangle_t$ is the time and ensemble average of the intensity.

For some general types of emission $p(I)$ is shown in Fig. 4. Nonfluctuating coherent light, i.e., an amplitude-stabilized light wave, is characterized by

$$p_{\mathrm{coh}}(I)=\delta(I-\langle I\rangle_t), \tag{9}$$

where δ denotes Dirac's delta function and $\langle I\rangle_t$ is the mean intensity. The multimode laser of independent modes shows even for a few modes the same intensity probability distribution as Gaussian light. This was

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[33] G. J. Troup, *Optical Coherence Theory—Recent Developments* (Methuen and Company Ltd., London, 1967), p. 22.

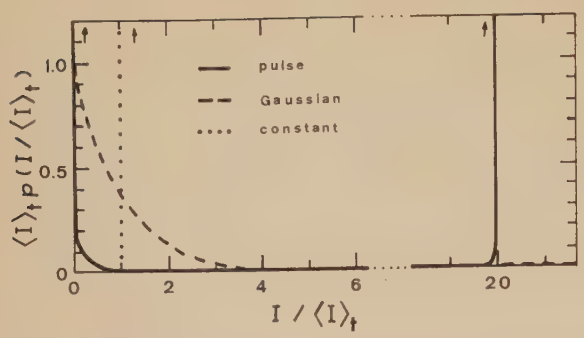

FIG. 4. Intensity probability distribution for an amplitude-stabilized light wave (constant), a Gaussian light source, and an ideally mode-locked laser emission (pulse).

first mentioned by Mandel and Wolf[1] and can be shown by use of the arguments of Glauber[34] to hold exactly for a large number of independently oscillating modes. From Glauber's derivation it can be seen that a random distribution of the phases of the modes within the interval $(0,2\pi)$ is the essential requirement for obtaining Gaussian light. In particular, the exact distributions of the real amplitudes (or intensities) of the modes are not important for the result, and only the sum of the mean intensities of the modes, which is equal to the mean intensity of the Gaussian emission, has to be known. For this reason, the real amplitudes of the modes can be assumed to have constant values in the context of the present paper without restriction of the generality.

The intensity fluctuations of thermal or Gaussian light are caused by the energy fluctuations of the emitting system in thermal equilibrium and are therefore described by a Boltzmann distribution:

$$p_G(I)=\langle I\rangle_t^{-1}\exp(-I/\langle I\rangle_t). \tag{10}$$

The intensity distribution $p_p(I)$ for a short pulse of Gaussian shape can be calculated from the time-dependent intensity

$$\begin{aligned} I_p(t)&=I_0\exp[-(t/\Delta t)^2]\\ &=(T/\Delta t)(\langle I\rangle_t/\sqrt{\pi})\exp[-(t/\Delta t)^2]\\ &\qquad(-\tfrac{1}{2}T\le t\le+\tfrac{1}{2}T). \end{aligned} \tag{11}$$

In Eq. (11), I_0 is the peak intensity, Δt is the width of the pulse, and $\langle I\rangle_t$ is the time average of the intensity if the observation time is restricted to the interval from $-\frac{1}{2}T$ to $+\frac{1}{2}T$. For a periodic train of pulses of similar shape, the interval T is equal to the separation of the pulses. With the relation[35]

$$p(I)=(1/T)\,|dI/dt|^{-1}, \tag{12}$$

the intensity distribution for a short pulse is obtained as

$$p_p(I)=\frac{\Delta t}{T}\frac{1}{I\sqrt{\ln(I_0/I)}},\quad I_0=\frac{T}{\Delta t}\frac{\langle I\rangle_t}{\sqrt{\pi}},\quad I(\tfrac{1}{2}T)\le I\le I_0. \tag{13}$$

In Fig. 4, the intensity distributions corresponding to the same mean intensity $\langle I\rangle_t$ for a short pulse with $T=36\Delta t$, for Gaussian emission (e.g., free-running multimode laser) and for constant amplitude coherent light (e.g., amplitude-stabilized FM laser[36]), are compared. In addition to these limiting cases, the intensity distributions for partial mode locking, with restricted phase variations as shown in Fig. 2, can be considered. An increase of the fluctuations of the phases of the modes results in a decrease of the probability of high intensities and an increase of medium intensities. A knowledge of the intensity probability distributions is the basis upon which to calculate the statistical distribution of the photons[27] and also to calculate the relative efficiency of multiphoton effects. This is first demonstrated for second-harmonic generation.

Second-harmonic generation is known to be proportional to the square of the instantaneous intensity, in so far as saturation effects may be neglected. The average intensity (time and/or ensemble average) of the second harmonic, $\langle\langle I_2\rangle\rangle_t$, is therefore given by

$$\langle\langle I_2\rangle\rangle_t\sim\langle\langle I^2\rangle\rangle_t=\int_0^\infty I^2p(I)dI. \tag{14}$$

This is recognized as the second moment of the intensity distribution $p(I)$. For the three discussed examples given in Eqs. (9), (10), and (13) the second moments are found to be

$$\begin{aligned} &\langle\langle I_{\rm coh}{}^2\rangle\rangle_t=\langle I\rangle_t{}^2;\quad \langle\langle I_G{}^2\rangle\rangle_t=2\langle I\rangle_t{}^2;\\ &\qquad\langle\langle I_p{}^2\rangle\rangle_t=(2\pi)^{-1/2}(T/\Delta t)\langle I\rangle_t{}^2. \end{aligned} \tag{15}$$

These results are in agreement with those obtained by rigorous calculations in the limit of an infinite number of modes. The higher moments of $p(I)$ for partial mode locking with phase fluctuations are calculated for a pulse width Δt with $T=36\Delta t$, corresponding to the numerically computed examples in Sec. II. The decrease of the second moment with increased phase interval ϕ is shown in Fig. 5.

Analogous to second-harmonic generation or two-photon effects, the average intensity of the nth harmonic, or generally n-photon effects, is proportional to the nth moment of the intensity distribution $p(I)$ of the fundamental light field. For the three discussed examples given in Eqs. (9), (10), and (13) the nth moments become

$$\begin{aligned} &\langle\langle I_{\rm coh}{}^n\rangle\rangle_t=\langle I\rangle_t{}^n,\quad \langle\langle I_G{}^n\rangle\rangle_t=n!\langle I\rangle_t{}^n,\\ &\langle\langle I_p{}^n\rangle\rangle_t=n^{-1/2}\pi^{-(n-1)/2}(T/\Delta t)^{n-1}\langle I\rangle_t{}^n. \end{aligned} \tag{16}$$

[34] R. J. Glauber, Phys. Rev. **131**, 2766 (1963), see particularly Eqs. (8.4) and (8.5).

[35] J. S. Bendat and A. G. Piersol, *Measurement and Analysis of Random Data* (John Wiley & Sons, Inc., New York, 1966), p. 61, Eq. (3.13).

[36] S. E. Harris, Appl. Opt. **5**, 1639 (1966).

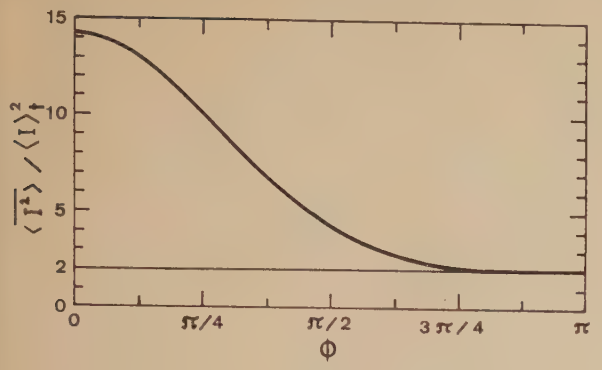

FIG. 5. Second moment of the intensity probability distribution of partially mode-locked emissions of 31 modes of amplitudes having a Gaussian envelope for phase angles φ randomly distributed in the interval $-\phi < \varphi < +\phi$.

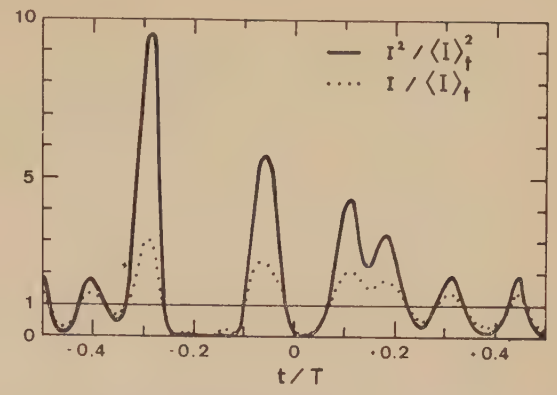

FIG. 6. Example of the instantaneous intensity of a laser with 31 modes of amplitudes having a Gaussian envelope and randomly distributed phase angles (···) compared with its second harmonic (——), both normalized with respect to the mean fundamental intensity $\langle I \rangle_t$.

It is seen from the above discussion that the knowledge of the moments of the intensity distribution is of particular interest for characterizing a light field with respect to its nonlinear interaction with matter. The above results also show that the higher moments of $p(I)$ of an unknown light field can be measured, e.g., by comparison of the average intensity of higher harmonics with the corresponding average intensity either of a light field having constant intensity or of a Gaussian light field of known mean intensity $\langle I \rangle_t$ with respect to the unknown light field.

As an example, the second moment of the second-harmonic light field generated by Gaussian light will be discussed. The instantaneous intensity I_2 of the second harmonic is proportional to the square of the instantaneous intensity I of the fundamental Gaussian light, with a proportionality factor γ including nonlinearity and geometry of the process. Therefore, the second moment of the intensity distribution of the second harmonic can be expressed by the fourth moment of the fundamental given in Eq. (16) by

$$\langle\langle I_2^2 \rangle\rangle_t = \gamma^2 \langle\langle I_G^4 \rangle\rangle_t = 4! \gamma^2 \langle I \rangle_t^4. \tag{17}$$

Using Eq. (15), the average intensity of the second harmonic is given by

$$\langle\langle I_2 \rangle\rangle_t = \gamma \langle\langle I_G^2 \rangle\rangle_t = 2! \gamma \langle I \rangle_t^2. \tag{18}$$

The second moment of the intensity distribution of the second harmonic generated by Gaussian light is therefore related to its average intensity by

$$\langle\langle I_2^2 \rangle\rangle_t = 6 \langle\langle I_2 \rangle\rangle_t^2. \tag{19}$$

This is different from the relation for the fundamental of Gaussian light given in Eq. (15).

Thus, the intensity probability distribution of the second-harmonic light field generated by Gaussian light is different from the distribution of the fundamental. This is obviously true because the higher intensities are enhanced by the second-harmonic generation with respect to the lower intensities. In illustration, the fluctuating second-harmonic intensity was numerically computed and is shown in Fig. 6, using an example from Fig. 1(a) for the fundamental.

By comparing the factors (2 and 6) in Eqs. (15) and (19) it becomes obvious that these intensity distributions are of central interest for nonlinear optics experiments. To show that they can be quite different for different creation mechanisms of light fields, especially if nonlinear optics interactions are involved, these properties shall be described in some detail.

IV. PROBABILITY DISTRIBUTION OF HIGHER HARMONICS

As shown at the end of the preceding chapter, the fluctuation behavior changes by the second-harmonic generation of a Gaussian source. As a consequence, the photon statistics and the intensity probability distribution $p(I)$ are no longer the same.

This fact can be understood by considering the different creation mechanisms of the various light fields. In a thermal light source the transition probability is given by a constant-transition matrix element for spontaneous emission multiplied by the instantaneous population of the upper level of the radiating transition. As this population fluctuates according to a Boltzmann distribution, the radiated intensity fluctuates in the same manner and corresponds to a Gaussian light field.

In stimulated emission the transition probability is given by the constant-transition matrix element for stimulated emission multiplied by the instantaneous population of the upper level of the radiating transition but now multiplied further by the instantaneous intensity of the stimulating light field inducing the transition. The process of optical mixing works in the same way. Here, the generated intensity is proportional to the population of the virtual level populated by one of the two primary light signals, multiplied by the intensity of the other light field. The instantaneous

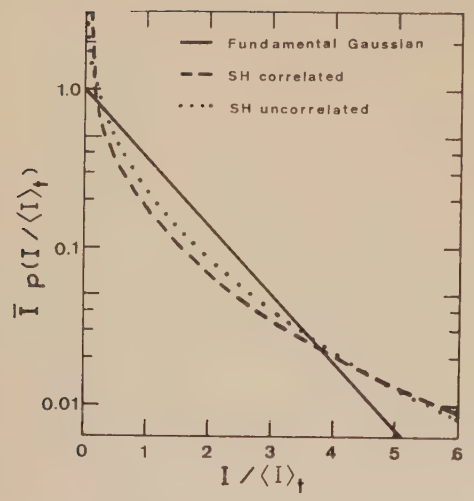

FIG. 7. Intensity probability distributions for a Guassian light wave, its second harmonic and the second harmonic of two uncorrelated Gaussian light waves. The last case corresponds to the optical mixing of two independent Gaussian light fields or to the optical mixing of a Gaussian light field with a delayed portion of itself with $\tau \gg \Delta\nu^{-1}$.

generated intensity is thus proportional to the product of the two primary instantaneous light fields, a well known result. As a consequence, the statistical behavior can be obtained by superimposing the two statistical distributions of the primary light fields. In general, the statistical behavior is changed by such superpositions. Some simple cases shall be discussed.

Note that this rule is only valid if saturation effects can be neglected and if the effects of smearing out in time between the different light fields, originating from dispersion and phasematching effects in thick crystals, can be neglected.[37–41]

In optical-mixing experiments it is necessary to distinguish carefully between the cases where the initial signals are correlated (dependent of each other) or not. To compute the probability distribution of the generated light we have to use different mathematical methods. For correlated signals the kind of correlation must be known exactly. Thus, in general, the cases of uncorrelated signals are easier to treat.

In the case of sum-frequency generation of two *independent* (*uncorrelated*) *signals* the probability distributions have, following the rules of probability calculations, to be multiplied in a suitable manner. This shall be demonstrated for the case of sum-frequency generation. Denoting, respectively, for the light signals I_1 and I_2 the probability distributions $p_1(I_1)$ and $p_2(I_2)$, the probability distribution $p_{12}(I_{12})$ of the generated light field becomes

$$p_{12}(I_{12}) = \int_0^\infty \int_0^\infty p_1(I_1) p_2(I_2) \delta(I_{12} - I_1 I_2) dI_1 dI_2, \quad (20)$$

where Dirac's delta function $\delta(I_{12}-I_1I_2)$ stands for the condition $I_{12}=I_1I_2$. Integration over the variable I_2 leads to

$$p_{12}(I_{12}) = \int_0^\infty p_1(I_1) p_2(I_{12}/I_1) I_1^{-1} dI_1. \quad (21)$$

If one of the two signals has a constant intensity, e.g., $I_1=I_0$, then p_1 becomes $p_1(I_1)=\delta(I_1-I_0)$ and then

$$p_{12}(I_{12}) = I_0^{-1} p_2(I_{12}/I_0). \quad (22)$$

The intensity probability distribution for the sum-frequency signal is of the same type as the distribution of the signal I_2. It is also seen immediately that in the trivial case of mixing two signals, both of constant intensity, again a field with constant intensity is obtained.

A case of special interest is the optical mixing of two independent Gaussian-type light fields with

$$p_1(I_1) = \langle I_1\rangle_t^{-1} \exp(-I_1/\langle I_1\rangle_t), \quad (23)$$

and

$$p_2(I_2) = \langle I_2\rangle_t^{-1} \exp(-I_2/\langle I_2\rangle_t). \quad (24)$$

Substitution of Eqs. (23) and (24) into (21) gives

$$p_{12}(I_{12}) = \langle I_{12}\rangle_t^{-1} \int_0^\infty I_1^{-1} \times \exp[-(I_1/\langle I_1\rangle_t + I_{12}\langle I_1\rangle_t/I_1\langle I_{12}\rangle_t)] dI_1, \quad (25)$$

which is an integral of the form

$$\int_0^\infty x^{-1} \exp[-(ax+bx^{-1})] dx = 2K_0(2(ab)^{1/2}), \quad (a, b>0) \quad (26)$$

where K_0 is a modified Bessel function of the second kind and 0th order,[42] resulting in

$$p_{12}(I_{12}) = 2\langle I_{12}\rangle_t^{-1} K_0(2(I_{12}/\langle I_{12}\rangle_t)^{1/2}). \quad (27)$$

In Fig. 7, the corresponding curve is given. Note that this result is independent of the bandwidth of the two Gaussian light fields and is also correct for the mixing of two Gaussian light fields with different bandwidths.

Whereas the preceding examples showed the mixing of uncorrelated signals, the process of second-harmonic generation is an example of the interaction of *correlated signals*, here a light signal interacts with itself. The instantaneous intensity of the second-harmonic light field is proportional to the squared instantaneous intensity of the fundamental. For an initial signal of

[37] J. Ducuing and J. A. Armstrong, in *Third Conference on Quantum Electronics*, edited by N. Bloembergen and P. Grivet (Dunod Cie., Paris, 1963), pp. 1643–1650.
[38] R. C. Miller, Phys. Letters **26A**, 177 (1968).
[39] J. Comly and E. Garmire, Appl. Phys. Letters **12**, 7 (1968).
[40] S. L. Shapiro, Appl. Phys. Letters **13**, 19 (1968).
[41] E. Mathieu and H. P. Weber, Z. Angew. Math. Phys. **19**, 980 (1968).
[42] See W. Gröbner and N. Hofreiter, *Integraltafeln—2. Teil: Bestimmte Integrale* (Springer-Verlag, Vienna, 1961), 3rd ed., Vol. 2, Eq. (314.9e).

constant intensity, this results again in a second-harmonic signal of constant intensity. For a short pulse of duration Δt we obtain again a short pulse, now of duration $\Delta t/\sqrt{2}$. Deviations from this result caused by bandwidth limitations[39,40] or group velocity effects[41] arising in thick crystals are discussed elsewhere.

The intensity distribution for second-harmonic light $p_{SH}(I_{SH})$ generated by a fundamental Gaussian light field is found from the Boltzmann distribution for the fundamental [see, e.g., Eq. (10)], by substituting $I=\sqrt{I_{SH}}$ and $dI=\frac{1}{2}I_{SH}^{-1/2}dI_{SH}$ and using the result of Eq. (15) for the average second-harmonic intensity. This yields

$$p_{SH}(I_{SH})dI_{SH}=(2\langle I_{SH}\rangle_t I_{SH})^{1/2}\times\exp[-(2I_{SH}/\langle I_{SH}\rangle_t)^{1/2}]dI_{SH}. \quad (28)$$

This result is plotted in Fig. 7 on a logarithmic scale, where the Boltzmann distribution of the fundamental is also given. It is seen that the probabilities of very low and very high intensities are enhanced, as was predicted by observation of Fig. 6. In Fig. 7 is given the distribution of the second harmonic generated by the interaction of a Gaussian field with a delayed portion of itself for delays $\tau\gg\Delta\nu^{-1}$. This situation is identical with the optical mixing of two independent Gaussian light fields discussed before, since the correlation time of Gaussian light is equal to the inverse bandwidth. It is observed in this case as well that the probability of very low and very high intensities is enhanced, but not as much as in the case without delay, which is identical to normal second-harmonic generation.

For optical-mixing and second-harmonic generation with signals of a defined time behavior, i.e., fixed phases of the modes, the two intensities have to be considered as completely correlated for all delay times τ. The generated intensity can be calculated by direct multiplication of the two time-dependent fundamental intensities. For example, in the case of a bandwidth-limited short pulse the intensities are obviously also correlated for $\tau\gg\Delta\nu^{-1}$. For partially mode-locked laser emission, i.e., residual phase fluctuations of the modes, there is also some correlation left for delay times $\tau\gg\Delta\nu^{-1}$, and therefore these cases can neither be treated as completely correlated nor as completely uncorrelated.

For the calculation of intensity probability distributions of higher-harmonic generation, the same formalism as for second-harmonic generation can be used. For the nth harmonic of a Gaussian light field this results in

$$p_n(I_n)dI_n=n^{-1}(n!)^{1/n}I_n^{(1/n-1)}\langle I_n\rangle_t^{-1/n}\times\exp[-(n!I_n/\langle I_n\rangle_t)^{1/n}]. \quad (29)$$

By a careful inspection of the formalism it is seen that it is irrelevant whether the nth harmonic is generated in one step or stepwise. In Sec. VI, it will be shown that only the order of the generated field (sum and difference frequency-generation processes can be included for the different steps as well) is of importance for the intensity probability distributions. Note that this conclusion is only valid as long as the correlation properties throughout the whole process are conserved.

V. INTENSITY-CORRELATION FUNCTIONS

The intensity correlations, also known as higher-order coherence functions, are of special interest because they can be obtained directly from measurements involving multiphoton effects. The various types of coincidence measurements may be described[19–21] by intensity-correlation functions. Experimental techniques giving second-[6,7,9,18] and third-[10]order correlations were used to measure the time duration of ultrashort light pulses, and experiments giving third-order correlations were proposed[28] to measure the shape asymmetry of ultrashort light pulses. Recently it was shown[43] that, from a knowledge of the second- and third-order correlations, even the exact shape of light pulses can be obtained, but obviously only in cases without statistical fluctuations. Additional information can be obtained by direct interpretation of the correlation functions, leading immediately to an evaluation of the normalized moment of the probability distribution of the same order. In addition, conclusions can be made regarding the behavior of light in time, e.g., the duration of fluctuations or pulses and repetition periods.

Normalized intensity-correlation functions $g^{(n)}$ are defined as

$$g^{(n)}(\tau_1,\tau_2,\cdots,\tau_{n-1})=\frac{G^{(n)}(\tau_1,\tau_2,\cdots,\tau_{n-1})}{\langle\langle I(t)\rangle\rangle_t^n}=\frac{\langle\langle I(t)I(t+\tau_1)I(t+\tau_2)\cdots I(t+\tau_{n-1})\rangle\rangle_t}{\langle\langle I(t)\rangle\rangle_t^n}. \quad (30)$$

First, the second-order correlations will be discussed to work out the fundamental properties of the intensity-correlation functions.

Second-Order Correlation

By use of Eq. (30), the normalized second-order intensity correlation $g^{(2)}(\tau)$ is found to be

$$g^{(2)}(\tau)=\frac{G^{(2)}(\tau)}{\langle\langle I(t)\rangle\rangle_t^2}=\frac{\langle\langle I(t)I(t+\tau)\rangle\rangle_t}{\langle\langle I(t)\rangle\rangle_t^2}. \quad (31)$$

Inspection of the correlation function shows that it has necessarily a maximum at $\tau=0$ for any light field and that $g^{(2)}(0)\geq 1$. It is also found that for increasing relative variation of the intensity $I(t)$ the value of $g^{(2)}(0)$ is a good measure of the amount of intensity variations and fluctuations. If the intensities for delay times τ' in an interval (τ_1,τ_2) are uncorrelated (statistically inde-

[43] E. I. Blount and J. R. Klauder, J. Appl. Phys. **40**, 2874 (1969).

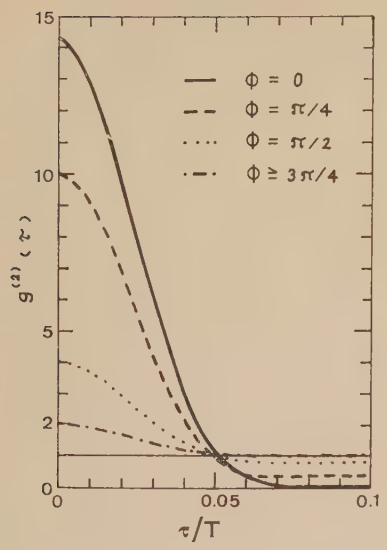

FIG. 8. Normalized intensity-correlation functions for different degrees of partial mode-locking. The phase angles are randomly distributed in the interval $-\phi<\varphi<+\phi$.

pendent) $g^{(2)}(\tau')$ becomes

$$g^{(2)}(\tau')=\frac{\langle\langle I(t)\rangle\rangle_t\langle\langle I(t+\tau')\rangle\rangle_t}{\langle\langle I(t)\rangle\rangle_t^2}\equiv 1. \tag{32}$$

Otherwise, $g^{(2)}(\tau)$ takes the value $0\leq g^{(2)}(\tau)\leq g^{(2)}(0)$.

For the special cases of light fields, introduced in Sec. III, a brief discussion shall be given of the second-order correlation functions. All of them are assumed to have the same spectral content. The successive change of $g^{(2)}(\tau)$ from the case of free-running laser emission (Gaussian light) to the case of ideal mode-locking by a stepwise reduction of the interval of the random-phase variations (partial mode-locking) is shown in Fig. 8. For Gaussian light ($\phi=\pi$), $g^{(2)}$ has the values $g^{(2)}(0)=2$ and $g^{(2)}(\tau\gg\Delta\nu^{-1})=1$, which was first proved experimentally by Hanbury-Brown and Twiss.[44] For a bandwidth-limited short pulse, resulting from ideal mode-locking, $g^{(2)}$ decreases from a value given by the observation time T and the pulse width $\Delta t=\Delta\nu^{-1}$ at $\tau=0$ to values near zero for $\tau\gg\Delta\nu^{-1}$. All curves show essentially the same width of the peak around $\tau=0$, given by the inverse bandwidth $\Delta\nu^{-1}$. Note that only for Gaussian light are the intensities completely uncorrelated for $\tau\gg\Delta\nu^{-1}$.

Only systematic deviation of the phases from the ideally mode-locked situation results, as demonstrated in Sec. III, in an increase of the pulse width and a decrease of the peak intensity. The same behavior is also found for $g^{(2)}$ with a peak value given by the observation time T and the lengthened pulse width $\Delta t>\Delta\nu^{-1}$ (Fig. 9). In this case, the width of the second-order correlation is given by the pulse duration Δt and not by the inverse bandwidth $\Delta\nu^{-1}$. For a coherent or intensity-stabilized light field, e.g., a single-mode or an FM laser, $g^{(2)}$ has the constant value $g^{(2)}(\tau)\equiv 1$. For laser emission $g^{(2)}(\tau)$ becomes periodic, because of the constant-difference frequency of the modes which results in a repetition of the intensity fluctuations and variations.

As seen from these examples the second-order correlation function contains essential information on the variations and fluctuations of the light intensity. It is well known[45] that the Fourier transformation of $G^{(2)}(\tau)$ is equal to the power spectrum $|\tilde{I}(\nu)|^2$ of the intensity of the light field,

$$|\tilde{I}(\nu)|^2=\int_{-\infty}^{+\infty}G^{(2)}(\tau)e^{-i2\pi\nu\tau}d\tau,\quad \tilde{I}(\nu)=\int_{-\infty}^{+\infty}I(t)e^{-i2\pi\nu t}dt. \tag{33}$$

Note that the spectrum $\tilde{I}(\nu)$ of the intensity has to be distinguished from the spectrum of the oscillating light wave itself, and that the power spectrum $|\tilde{I}(\nu)|^2$, and therefore also $G^{(2)}(\tau)$, does not contain the phase information. Thus, the knoweldge of $G^{(2)}(\tau)$ is not sufficient to evaluate the time behavior of the light intensity.[46]

In the following is evaluated the connection between the intensity correlation and the intensity probability $p(I)$, their moments and the efficiency of multiphoton effects. The numerator $G^{(2)}(\tau)$ in Eq. (31) becomes for $\tau=0$

$$G^{(2)}(0)=\langle\langle I^2\rangle\rangle_t, \tag{34}$$

which is exactly the second moment of $p(I)$, as defined in Eq. (14). The averaged relative efficiency of second-harmonic generation of a given light field with respect to nonfluctuating light, as first introduced by Ducuing and Bloembergen,[2] is equal to the normalized second-order moment

$$g^{(2)}(0)=\langle\langle I^2\rangle\rangle_t/\langle\langle I\rangle\rangle_t^2. \tag{35}$$

This normalization can be obtained from integration of $G^{(2)}(\tau)$ over an interval of the delay time τ corresponding to the observation time T:

$$\frac{1}{T}\int_0^T G^{(2)}(\tau)d\tau=\frac{1}{T}\int_0^T\langle\langle I(t)I(t+\tau)\rangle\rangle_t d\tau. \tag{36}$$

Interchanging integration and averaging processes yields the expression

$$\frac{1}{T}\int_0^T G^{(2)}(\tau)d\tau=\langle\langle I\rangle\rangle_t^2=\langle I\rangle_t^2, \tag{37}$$

as long as the time average of the intensity is considered to be constant within the ensemble. Therefore, the

[44] R. H. Brown and R. Q. Twiss, Proc. Roy. Soc. (London) A242, 300 (1957); A243, 291 (1958).

[45] L. Levi, *Applied Optics* (John Wiley & Sons, Inc., New York, 1968), Vol. I, pp. 138–140.

[46] H. Bebié (private communication).

intensity correlation $g^{(2)}(\tau)$, and obviously also $g^{(2)}(0)$, can be calculated from the knowledge of $G^{(2)}(\tau)$ using the equation

$$g^{(2)}(\tau)=G^{(2)}(\tau)\left[\frac{1}{T}\int_0^T G^{(2)}(\tau)d\tau\right]^{-1}. \quad (38)$$

As shown elsewhere[20] and reviewed in Sec. VI, $G^{(2)}(\tau)$ is proportional to the mean intensity of the second-harmonic light produced by two parts of the fundamental light field with a relative delay τ.

Higher-Order Correlation

Similar considerations can also be made for higher-order correlations. Note that they depend on several delay times. The general properties shall be discussed for the third-order correlation and then be generalized to the nth order. $G^{(3)}(\tau_1,\tau_2)$ is defined as

$$G^{(3)}(\tau_1,\tau_2)=\langle\langle I(t)I(t+\tau_1)I(t+\tau_2)\rangle\rangle_t. \quad (39)$$

Obviously the third-order moment of the intensity probability distribution is obtained for $\tau_1=\tau_2=0$ as

$$G^{(3)}(0,0)=\langle\langle I^3\rangle\rangle_t. \quad (40)$$

The value of the correlation for $(\tau_1, \tau_2\neq 0)$ depends strongly on the time behavior of the intensity. If the intensity has a defined time behavior without statistical fluctuations, the knowledge of $G^{(3)}(\tau_1,\tau_2)$ and $G^{(2)}(\tau)$ is sufficient to describe the shape of the intensity completely, as recently demonstrated by Blount and Klauder.[43] In the case of statistically fluctuating intensities, such as Gaussian light, for sufficiently large relative delays, the intensities can be considered uncorrelated. Assume a Gaussian light field $G^{(3)}$ can be calculated. For $\tau_1=0$ and $\tau_2\gg\Delta\nu^{-1}$ Eq. (40) yields with (15)

$$G^{(3)}(0,\tau_2)=\langle\langle I(t)I(t)\rangle\langle I(t+\tau_2)\rangle\rangle_t=\langle\langle I^2\rangle\rangle_t\langle I\rangle_t=2\langle I\rangle_t^3, \quad (41)$$

and finally $G^{(3)}$ becomes for τ_1, τ_2, $|\tau_1-\tau_2|\gg\Delta\nu^{-1}$

$$G^{(3)}(\tau_1,\tau_2)=\langle\langle I(t)\rangle\langle I(t+\tau_1)\rangle\langle I(t+\tau_2)\rangle\rangle_t=\langle I\rangle_t^3. \quad (42)$$

The normalization of $G^{(3)}$ is found in a similar way as that shown for $G^{(2)}$, namely,

$$g^{(3)}(\tau_1,\tau_2)=G^{(3)}(\tau_1,\tau_2)\left[\frac{1}{T^2}\int\int_0^T G^{(3)}(\tau_1,\tau_2)d\tau_1 d\tau_2\right]^{-1}. \quad (43)$$

For higher-order intensity correlations, similar relations are obtained for the nth-order moment of the intensity probability distribution $p(I)$ and for the normalized correlation function $g^{(n)}(\tau_1,\cdots,\tau_{n-1})$.

As examples, Table I gives special numerical values of the three lowest-order correlation functions for a light source of the Gaussian type. These values are obtained in the same manner as that used before to derive Eqs. (41) and (42), i.e., by decomposition of the product of the intensities into uncorrelated groups of correlated intensities.

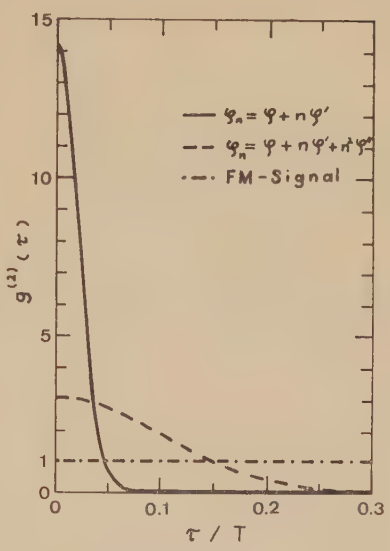

FIG. 9. Normalized intensity-correlation functions for fixed-phase angles with systematic deviation from ideal mode-locking and the extreme case of an FM laser output.

VI. MEASUREMENT OF CORRELATION FUNCTIONS AND HIGHER-ORDER MOMENTS

To describe the optically nonlinear processes of sum and difference frequency generation it is convenient to use the complex analytic signals $V(t)$ associated with the real electric field $E(t)$.[47] The induced nonlinear polarizations P^{NL} are for sum and difference frequency generation[48] of the two signals V_1 and V_2:

$$P_{sum}{}^{NL}=\chi_{1+2}{}^{NL}V_1V_2, \quad (44)$$

and

$$P_{diff}{}^{NL}=\chi_{1-2}{}^{NL}V_1V_2^*, \quad (45)$$

where χ^{NL} are the appropriate nonlinear constants, and V^* denotes the complex conjugate of V. Using suitable proportionality factors α and β, the analytic signals

TABLE I. Numerical values of normalized higher-order correlation functions for a Gaussian fundamental light field.

	$g^{(2)}(\tau_1)$	$g^{(3)}(\tau_1,\tau_2)$	$g^{(4)}(\tau_1,\tau_2,\tau_3)$
$\tau_1=\tau_2=\tau_3=0$	$2!=2$	$3!=6$	$4!=24$
$\tau_1\gg\Delta\nu^{-1}$, $\tau_2=\tau_3=0$	$(1!)^2=1$	$(1!)(2!)=2$	$(3!)(1!)=6$
$\tau_1=\tau_2\gg\Delta\nu^{-1}$, $\tau_3=0$	···	$(2!)(1!)=2$	$(2!)^2=4$
$\tau_1, \tau_2\gg\Delta\nu^{-1}$, $\tau_3=0$ $\lvert\tau_1-\tau_2\rvert\gg\Delta\nu^{-1}$	···	$(1!)^3=1$	$(1!)^2(2!)=2$
$\tau_1, \tau_2, \tau_3\gg\Delta\nu^{-1}$ $\lvert\tau_1-\tau_2\rvert, \lvert\tau_1-\tau_3\rvert, \lvert\tau_2-\tau_3\rvert\gg\Delta\nu^{-1}$	···	···	$(1!)^4=1$

[47] W. H. Steel, *Interferometry* (Cambridge University Press, New York, 1967), pp. 22–23.

[48] J. A. Armstrong, N. Bloembergen, J. Ducuing, and P. S. Pershan, Phys. Rev. 127, 1918 (1962).

emitted by these nonlinear polarization sources are given by

$$V_{\text{sum}}=\alpha P_{\text{sum}}{}^{\text{NL}}=\alpha\chi_{1+2}{}^{\text{NL}}V_1V_2 \tag{46}$$

and

$$V_{\text{diff}}=\beta P_{\text{diff}}{}^{\text{NL}}=\beta\chi_{1-2}{}^{\text{NL}}V_1V_2{}^{*}. \tag{47}$$

These relations are valid only under the following assumptions:

(1) The fundamental light fields V_1 and V_2 are attenuated only by a neglectable amount.[37]

(2) The total spectra contained in V_1 and V_2 contribute uniformly to sum or difference frequency generation.

While the first assumption can be fulfilled by considering weak fields V_1 and V_2, it is a serious problem to fulfill the second one. For thick crystals,[37–41] the phase-matching condition to produce the sum frequency is not fulfilled for all interacting modes. Further, the group velocities of the different frequencies are different. The first point diminishes the spectrum, and therefore the duration of the fluctuations is increased; the second one averages out the maxima of the fluctuations, e.g., in the case of second-harmonic generation.

The intensity I_{sum} of the generated sum-frequency part becomes, using the relation $I=V\,V^{*}$ twice,

$$\begin{aligned}I_{\text{sum}}=V_{\text{sum}}V_{\text{sum}}{}^{*}&=(\alpha\chi_{1+2}{}^{\text{NL}})^2V_1V_2V_1{}^{*}V_2{}^{*}\\&=(\alpha\chi_{1+2}{}^{\text{NL}})^2I_1I_2.\end{aligned} \tag{48}$$

This relation will be used to calculate the intensity of the sum frequency generated by a signal $I(t)$ and the corresponding delayed signal $I(t+\tau)$. The above results are valid only if the sum frequency is generated by interaction of the two signals $I(t)$ and $I(t+\tau)$ and if one signal alone generates no second-harmonic light. Thus, such second-harmonic experiments require geometries in which the two primary signals can be distinguished. In the known experiments, this was realized by different directions[18] or polarizations[6–8,15] of the primary signals.

The instantaneous intensity I_{sum} of the generated signal is normally not accessible to a measurement, since it fluctuates at a rate similar to the fundamental. The measured intensities are mean values over the resolution time of the detection system,[2] a time usually of no significance compared to times of the light signal.

A more significant mean value is obtained by averaging over a time interval short enough to assure constant amplitudes and phases of the different modes, but long enough to smooth out the lowest beat frequency of the modes. These are the necessary conditions to measure a time average as defined in Sec. II, and it can therefore be written as

$$\langle I_{\text{sum}}\rangle_t(\tau)=(\alpha\chi_{1+1}{}^{\text{NL}})^2\langle I(t)I(t+\tau)\rangle_t, \tag{49}$$

which is proportional to the autocorrelation of the original signal $I(t)$.

In cases with statistical fluctuations, the same time-dependent intensity $I(t)$ cannot be reproduced. For this reason, it is more interesting to have the knowledge of the average behavior of the measured quantities with respect to the ensemble. The individually measured values for distinct samples of the ensemble fluctuate around this ensemble-averaged value. The ensemble-averaged mean intensity, as given in Eq. (49), is found to be proportional to the second-order intensity-correlation function, defined in Sec. V:

$$\begin{aligned}\langle\langle I_{\text{sum}}\rangle\rangle_t(\tau)&=(\alpha\chi_{1+1}{}^{\text{NL}})^2\langle\langle I(t)I(t+\tau)\rangle\rangle_t\\&=(\alpha\chi_{1+1}{}^{\text{NL}})^2G^{(2)}(\tau).\end{aligned} \tag{50}$$

This ensemble averaging is done in the experiment by considering[2,3] several independent emissions of the laser, such as different flashes or different spikes, or off-axial modes of different order, because the different modes are not stable relative to each other since amplitudes as well as relative phases are not stable in time.

It is seen that by such special experiments of second-harmonic generation an expression can be obtained which is proportional to the correlation function $G^{(2)}(\tau)$. By use of Eq. (38), the normalized correlation function $g^{(2)}(\tau)$ can be evaluated. It is of special interest to note that the normalized function is obtained from only one experiment and that additional measurements with lasers of exactly known characteristics are not necessary for calibration. The same consideration is obviously also valid for the second moment of $p(I)$. These moments are usually calibrated by comparing the averaged second-harmonic intensity of free-running and mode-locked laser operation. Such measurements have been used to show that lasers were completely mode-locked.[25,26]

Note that the relations to obtain $g^{(2)}$ are simple only in the case in which the second harmonic is generated by interaction of the two light fields alone. If one primary signal itself also generates second harmonic light, the correlation is superimposed on a background. Such results are also obtained in the experiments using two-photon absorption induced fluorescence[9] as the nonlinear effect. It was shown[19–22] that in this case the intensity of the fluorescence trace $F(\tau)$ can be described by

$$F(\tau)\sim G^{(2)}(0)+2G^{(2)}(\tau), \tag{51}$$

where the delay τ appears in the experiment as the space coordinate.

Similar results as for the sum frequency generation are obtained for difference frequency generation, since the expression for the intensity of the difference frequency

$$\begin{aligned}I_{\text{diff}}=V_{\text{diff}}V_{\text{diff}}{}^{*}&=(\beta\chi_{1-1}{}^{\text{NL}})^2V_1V_2{}^{*}V_1{}^{*}V_2\\&=(\beta\chi_{1-1}{}^{\text{NL}})^2I_1I_2\end{aligned} \tag{52}$$

takes the same final form as Eq. (48). Setting again $I_1=I(t)$ and $I_2=I(t+\tau)$ and averaging, with the same assumptions as above, $\langle\langle I_{\text{diff}}(\tau)\rangle\rangle_t$ also becomes pro-

portional to the second-order intensity correlation. In such an experiment it would be necessary to measure the power of the electric field generated by the optical "rectifying" process,[49,50] in which the field fluctuates with the highest beat frequency of the primary light field, i.e., with the inverse optical bandwidth. Such an experiment would be much more complicated to perform than measurements of the average intensity of the second harmonic, since bandwidths of 10^{12} Hz as emitted by solid-state lasers are quite common in optics but not in electronics.

In the same way as we have calculated second-order intensity correlations, higher orders may be obtained. The nonlinear polarization induced by two signals V_1 and V_2, where V_2 itself is generated as sum frequency of V_3 and V_4, is obtained as

$$P_{\text{sum}}{}^{\text{NL}}=\chi_{1+2}{}^{\text{NL}}V_1V_2=\chi_{1+2}{}^{\text{NL}}\chi_{3+4}{}^{\text{NL}}\alpha_{3+4}V_1(V_3V_4) \quad (53)$$

and

$$P_{\text{diff}}{}^{\text{NL}}=\chi_{1-2}{}^{\text{NL}}V_1{}^*V_2=\chi_{1-2}{}^{\text{NL}}\chi_{3+4}{}^{\text{NL}}\alpha_{3+4}V_1{}^*(V_3V_4). \quad (54)$$

The intensities of sum and difference frequencies generated by these two nonlinear polarizations can be calculated in a similar way as before:

$$\begin{aligned} I_{\text{sum}}&=(\chi_{1+2}{}^{\text{NL}}\chi_{3+4}{}^{\text{NL}}\alpha_{1+2}\alpha_{3+4})^2V_1(V_3V_4)V_1{}^*(V_3{}^*V_4{}^*)\\ &=(\chi_{1+2}{}^{\text{NL}}\chi_{3+4}{}^{\text{NL}}\alpha_{1+2}\alpha_{3+4})^2I_1I_3I_4. \end{aligned} \quad (55)$$

$$\begin{aligned} I_{\text{diff}}&=(\chi_{1-2}{}^{\text{NL}}\chi_{3+4}{}^{\text{NL}}\beta_{1-2}\alpha_{3+4})^2V_1{}^*(V_3V_4)V_1(V_3{}^*V_4{}^*)\\ &=(\chi_{1-2}{}^{\text{NL}}\chi_{3+4}{}^{\text{NL}}\beta_{1-2}\alpha_{3+4})^2I_1I_3I_4. \end{aligned} \quad (56)$$

Again, the results for sum and difference frequency generation are of the same form. For the situation $I_1=I(t)$, $I_3=I(t+\tau_1)$ and $I_4=I(t+\tau_2)$, with the same considerations for averaging as for the second-order correlation function, the average intensities $\langle\langle I_{\text{sum}}\rangle\rangle_t$ and $\langle\langle I_{\text{diff}}\rangle\rangle_t$ are found [see Eq. (50)] to be proportional to the third-order correlation

$$\begin{aligned} G^{(3)}(\tau_1,\tau_2)&=\langle\langle I(t)I(t+\tau_1)I(t+\tau_2)\rangle\rangle_t\\ &\propto\langle\langle I_{\text{sum}}(\tau_1,\tau_2)\rangle\rangle_t\\ &\propto\langle\langle I_{\text{diff}}(\tau_1,\tau_2)\rangle\rangle_t. \end{aligned} \quad (57)$$

It is seen also that $G^{(3)}$ is obtained by either sum or difference frequency generation. For difference frequency generation, relation (57) is only correct if the generated field, having necessarily the same frequency as the original signal, can be distinguished from the primary ones. It must be distinguishable, e.g., by propagation direction or polarization. In Fig. 10, an example of an experimental arrangement to measure $G^{(3)}(\tau_1,\tau_2)$ is given, using as a final signal the third harmonic. Recently, it was shown[28] that with two-photon absorption induced fluorescence by a signal and its second harmonic $G^{(3)}$ can also be obtained.

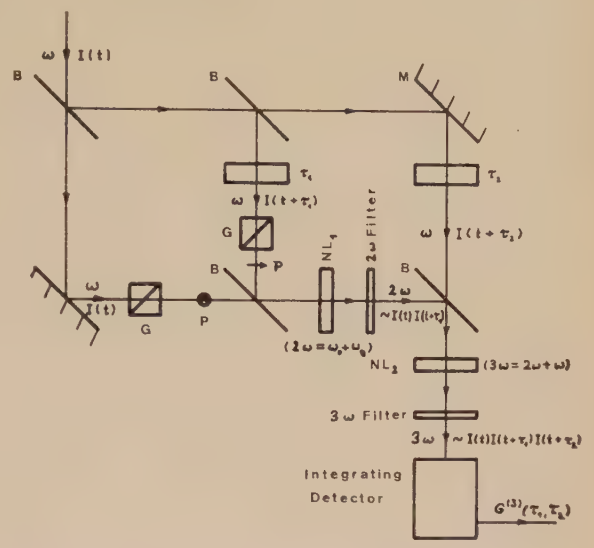

FIG. 10. Arrangement for measuring $G^{(3)}(\tau_1,\tau_2)$. B, beamsplitters; M, mirrors; τ_1, τ_2, optical delay units; G, linear polarizers with polarization directions P; NL_1, nonlinear crystal for optical mixing of two orthogonally linear polarized beams of the same frequency (see Ref. 7); NL_2, nonlinear crystal for optical mixing.

Higher-order correlation can be obtained by repeating the procedure shown above. It is interesting to notice that the same results are observed by considering sum or difference frequency generation. This is also true for the probability distributions of higher harmonics. The order of the effect is the important number, and to this number sum and difference frequency effects contribute in the same manner. For example, the intensity fluctuations of the optical rectification induced by a second-harmonic signal are identical with the intensity fluctuations of the fourth-harmonic signal, generated by double second-harmonic production. As a consequence, it is seen that in order to obtain the higher-order moments, measurements at progressively shorter wavelengths do not necessarily have to be carried out.

VII. CONCLUSIONS

Besides ideal mode-locking (resulting in bandwidth-limited short pulses) and completely independent oscillating modes (giving an emission which fluctuates as a Gaussian light field), intermediate cases of nonideal mode-locking have been discussed in detail.

The emission of lasers with modes having phase angles all distributed randomly in given intervals were considered. It was shown that by increasing the intervals the intensity of the pulses slowly decreases and the fluctuating background increases. The duration of the pulses remains essentially constant, namely, equal to the inverse of the oscillating bandwidth. A systematic deviation of the phase angles, at least quadratic with the mode number, is necessary for lengthening the pulses. It was shown that such quadratic deviations may

[49] M. Bass, P. A. Franken, J. F. Ward, and G. Weinreich, Phys. Rev. Letters 9, 446 (1962).
[50] M. Bass, P. A. Franken, and J. F. Ward, Phys. Rev. 138, A534 (1965).

be caused by dispersion effects corresponding to a quadratic dependence of the refractive index on the vacuum wavelength.

It was possible to explain by a superposition of these two effects all experimental results published so far, showing background and pulses longer than predicted by the total oscillating bandwidth.

The different light fields were described by the intensity probability distributions $p(I)$. They contain the averaged information on the light field, ensemble- as well as time-averaged. The average intensity in higher-harmonic generation experiments is proportional to the same order moment of $p(I)$, which moments describe also the higher degrees of coherence. By measurements of averaged intensities of higher harmonics the moments of light fields with a total bandwidth corresponding to a coherence length or a coherence time that could not be resolved by electronic devices are obtainable.

Further, it was shown that the intensity probability distributions are generally changed by optical mixing and also by higher-harmonic generation. The facts were worked out for second harmonic generation of a Gaussian light field and also for optical mixing of two correlated or uncorrelated Gaussian light fields. In both cases the fluctuations are more pronounced than in the Gaussian case.

In addition, the description of light fields by intensity-correlation functions was introduced. They describe the efficiency in higher-harmonic generation and optical-mixing processes for correlated primary light signals which have relative delays. They can be experimentally measured also for large bandwidths. The intensity correlations of a given order for a vanishing delay are equal to the same order moment of the intensity probability distribution $p(I)$. It was shown that from the knowledge of the whole nth-order correlation function $G^{(n)}$ the determination of the normalized nth moment of $p(I)$ is possible. The knowledge of the exact shape of $G^{(n)}$ contains even more information than the corresponding moment of $p(I)$. Especially from the shape of the correlation functions, information on very fast intensity variations can be obtained.

The efficiency of higher-harmonic generation for a given light field is determined by the corresponding higher-order moments of the intensity probability distribution. If in more general optical-mixing experiments, relative delays or different group velocities of the interacting parts of the light field become important, the intensity-correlation functions are necessary to describe the efficiency instead of the moments of $p(I)$. Also, in parametric amplifiers and oscillators or in stimulated scattering processes (as, for example, to investigate the forward-backward asymmetry in the stimulated Raman scattering), the intensity-correlation functions may be very helpful for the calculation of the conversion efficiency for different light fields.

ACKNOWLEDGMENTS

We are grateful to Professor K. P. Meyer for the interest he has taken in this work. Thanks are also due to Dr. R. F. Broom for assistance with the manuscript.

APPENDIX A

In Sec. II, the ensemble-averaged intensity $\langle I(t)\rangle$ of a multimode laser is calculated, assuming that the modes have phases φ_k that are randomly distributed within a fixed interval $-\phi<\varphi_k<+\phi$ all about the same center value $\varphi^0=0$. The calculation of $\langle I(t)\rangle$ shall be generalized in two ways:

(1) The center values of the phases $\varphi_k{}^0$ are allowed to be different for different modes.

(2) The phases are not randomly distributed around the center values $\varphi_k{}^0$, but have the normalized phase-distribution functions $p_k(\varphi_k)$ around $\varphi_k{}^0$. It will be assumed that these functions are of identical shape, so that

$$p_k(\varphi_k)=p(\varphi_k-\varphi_k{}^0)=p(\varphi)\quad\varphi=\varphi_k-\varphi_k{}^0. \quad \text{(A1)}$$

The instantaneous intensity $I(t)$ of a multimode laser is given by

$$I(t)=\sum_{n,m} a_n a_m \exp\{i[(n-m)\Omega t+(\varphi_n-\varphi_m)]\}, \quad \text{(A2)}$$

and its ensemble average becomes

$$\langle I(t)\rangle=\sum_n a_n{}^2+\sum_{n\neq m} a_n a_m \int\int_0^{2\pi} p_n(\varphi_n)p_m(\varphi_m) \times\exp\{i[(n-m)\Omega t+(\varphi_n-\varphi_m)]\}d\varphi_n d\varphi_m, \quad \text{(A3)}$$

since summation and averaging process are interchangeable and φ_n and φ_m are statistically independent for $n\neq m$. Using $\varphi_k=\varphi_k{}^0+\varphi$, this becomes

$$\langle I(t)\rangle=\sum_n a_n{}^2 +\sum_{n\neq m} a_n a_m \exp\{i[(n-m)\Omega t+(\varphi_n{}^0-\varphi_m{}^0)]\} \times\int\int_0^{2\pi} \exp\{i(\varphi-\varphi')\}p(\varphi)p(\varphi')d\varphi d\varphi'. \quad \text{(A4)}$$

In this equation φ and φ' are independent variables and the twofold integral can therefore be written as the product of two integrals, which are recognized as Fourier transformations $\tilde{p}(x)$ of $p(\varphi)$ at $x=1$. Equation (A4) therefore becomes

$$\langle I(t)\rangle=\sum_n a_n{}^2+|\tilde{p}(1)|^2\sum_{n\neq m} a_n a_m \times\exp\{i[(n-m)\Omega t+(\varphi_n{}^0-\varphi_m{}^0)]\}. \quad \text{(A5)}$$

This equation can be written in the following way:

$$\langle I(t)\rangle = [1-|\tilde{p}(1)|^2]\sum_n a_n{}^2 + |\tilde{p}(1)|^2 \times \sum_{n,m} a_n a_m \exp\{i[(n-m)\Omega t + (\varphi_n{}^0 - \varphi_m{}^0)]\}$$

$$= [1-|\tilde{p}(1)|^2]\langle I\rangle_t + |\tilde{p}(1)|^2 I_{\text{det}}(t), \qquad \text{(A6)}$$

which is a linear combination of the mean intensity $\langle I\rangle_t$ and the time-dependent determined intensity $I_{\text{det}}(t)$, arising if the fluctuations vanish, i.e., if all phases $\varphi_k = \varphi_k{}^0$.

As an example, the center phase angles $\varphi_k{}^0$ may have the values $\varphi_k{}^0 = \varphi + k\varphi' + k^2\varphi''$ according to the example of Sec. II to demonstrate the lengthening of optical pulses. With statistical fluctuations around these fixed values it is shown by Eq. (A6) that the ensemble-averaged intensity is the superposition of this lengthened pulse and a constant background.

APPENDIX B

The influence of a dispersive medium on the emission of an ideally mode-locked laser is investigated and discussed. It results in a systematic deviation of the phases with increasing mode number. The propagation of the light in the z direction is described by the wave-number k in the medium. For an ideally mode-locked pulse at $z=0$ the analytical signal $V(t,z)$ at position $z>0$ becomes

$$V(t,z) = \sum_{n=-N}^{+N} a_n \exp\{i[(\omega_0 + n\Omega)t - k_n z]\} = \sum_{n=-N}^{+N} a_n \exp\{i[(\omega_0 + n\Omega)t + \varphi_n]\}, \qquad \text{(B1)}$$

a signal with the same mode intensities but changed phases φ_n.

The phases φ_n can be calculated by expanding the wave number k around the center frequency ω_0 with respect to the frequency difference $(n\Omega)$ of the nth mode:

$$\varphi_n = -z\left[k_0 + \frac{dk}{d\omega}\bigg|_{\omega_0} n\Omega + \frac{1}{2}\frac{d^2k}{d\omega^2}\bigg|_{\omega_0} n^2\Omega^2 + \cdots\right]. \qquad \text{(B2)}$$

This expansion is a power series with respect to the mode number n counted from the center ω_0:

$$\varphi_n = \varphi + n\varphi' + n^2\varphi'' + \cdots, \qquad \text{(B3)}$$

as assumed for Eq. (7) in Sec. II. The linear term φ' and the second-order term φ'' are recognized as

$$\varphi' = \frac{-z\Omega}{v} \quad \text{and} \quad \varphi'' = -\frac{z\Omega^2}{2}\frac{d}{d\omega}\left(\frac{1}{v}\right), \qquad \text{(B4)}$$

where the group velocity $v = d\omega/dk$ is introduced. Therefore, a frequency-independent group velocity v results in a linear deviation of the phases with mode number only. Pulse lengthening, i.e., nonvanishing φ'', is caused by the frequency dependence of the group velocity, a well-known fact in radar engineering.

Usually a dispersive optical medium is described by its index of refraction η, rather than by the frequency dependence of the wave number. It turns out to be more convenient, for physical interpretation, to expand the refractive index around the center frequency with respect to the vacuum wavelength λ than with respect to the frequency ω. The phases φ_n are therefore

$$\varphi_n = -\frac{(\omega_0 + n\Omega)z}{c}\left[\eta_0 + \frac{d\eta}{d\lambda}\bigg|_{\omega_0}\Delta\lambda_n + \frac{1}{2}\frac{d^2\eta}{d\lambda^2}\bigg|_{\omega_0}(\Delta\lambda_n)^2 + \cdots\right]. \qquad \text{(B5)}$$

With $\lambda = 2\pi c/\omega$ the wavelength difference $\Delta\lambda_n$ of the nth mode to the center mode at ω_0 is found to be

$$\Delta\lambda_n = \lambda_n - \lambda_0 = -\frac{2\pi c}{\omega_0}\frac{n\Omega}{(\omega_0 + n\Omega)}. \qquad \text{(B6)}$$

Using Eqs. (B5) and (B6), the phases φ_n are given by the following power series with respect to the mode number n:

$$\varphi_n = -\frac{\omega_0 z}{c}\eta_0 - n\left[\frac{\Omega z}{c}\eta_0 - \frac{2\pi\Omega z}{\omega_0}\frac{d\eta}{d\lambda}\right] - n^2\left[\frac{2\pi^2\Omega^2 cz}{\omega_0{}^3}\frac{d^2\eta}{d\lambda^2}\right] + \cdots. \qquad \text{(B7)}$$

The comparison of Eqs. (B3) and (B7) shows that linear dispersion of the refractive index η with respect to the vacuum wavelength λ, i.e., $d^2\eta/d\lambda^2 = 0$, results only in a linear deviation φ' of the phases with mode number, and hence does not increase the initial pulse width. A further comparison of Eqs. (B3), (B4), and (B7) indicates that the difference between the group velocity v and the phase velocity c/η_0 is caused by the linear dispersion $d\eta/d\lambda$, and that the frequency dependence of the group velocity is only introduced by the second-order dispersion $d^2\eta/d\lambda^2$ and higher-order terms.

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IEEE JOURNAL OF QUANTUM ELECTRONICS
Vol. QE-6, No. 11, November 1970
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Statistical Amplitude and Phase Variations in Mode-Locked Lasers

RENÉ DÄNDLIKER, A. A. GRÜTTER, AND HEINZ P. WEBER

bstract—**The influence of statistical amplitude or phase vari-ns in the modes of a mode-locked laser on the time-dependent ut intensity and on the relative efficiency of nonlinear optical esses is investigated. The ensemble-averaged intensity, the -dependent distribution of the resulting complex amplitude, and second-order moment of the intensity distribution are calculated and discussed for both types of variations. As long as no systematic amplitude or phase changes are introduced, the pulse shape is maintained, but superimposed on a fluctuating background, and the efficiency for nonlinear optical processes is reduced. For random phase fluctuations over 2π the pulse behavior disappears completely and the output becomes Gaussian or chaotic light.**

Manuscript received January 19, 1970; revised May 15, 1970.
. A. Grütter is with the Institute of Applied Physics, University Berne, Berne, Switzerland.
. Dändliker was with the Institute of Applied Physics, University erne, Berne, Switzerland. He is now with Brown Boveri Research ter, Baden, Switzerland.
. P. Weber was with the Institute of Applied Physics, University erne. He is now with Bell Telephone Laboratories, Inc., Holmdel, .

Introduction

IT IS a well-known fact that the modes of the whole spectrum of a multimode laser may be locked, so that the laser output consists of a periodic train of short light pulses [1]. For ideal mode locking, i.e., all modes having equal phases and time-independent fixed

amplitudes with a smooth envelope function, the duration of the short pulses is equal to the inverse bandwidth of the total laser spectrum. Consecutive pulses are separated by a time period equal to the inverse mode spacing.

Recent experiments show deviations from this ideal behavior, namely, pulses that are longer [2] than the inverse oscillating bandwidth and also pulses that are superimposed on a background [3]–[6]. Several explanations for this behavior of partial mode locking can be given. The effects if only a part of the spectrum is locked [7] and the rest is free running as well as the influence of statistical [8], [9] and systematic [9], [10] deviations of the phase angles from the ideal values were investigated. It was shown especially that the statistical variations of the phase angles introduce a fluctuating background and the duration of the pulses remains the same, whereas systematic deviations of the phase angles are necessary to lengthen the pulses.

An alternate approach [11] to describe longer pulses is a model where in the spectrum, different domains are ideally locked to form short pulses that have no phase relation between each other. The superposition of such pulses that arrive essentially simultaneously results in pulses longer than the inverse oscillating bandwidth but having a pronounced substructure [12].

It is the purpose of this paper to investigate the influence of slowly varying statistical amplitude variations. Such variations originate from competition of the different modes as well as from the different Q values of the optical resonator for different modes. Systematic variations, leading to mode selection, are not considered. The effect of statistical amplitude variations is compared with the effect of statistical phase variations. In the first part, numerically computed examples of the time-dependent intensity and the analytically calculated ensemble-averaged intensities are compared. In the second part, these results are confirmed by a more comprehensive discussion based on the intensity distribution $p(I)$ and its moments, which determine the relative efficiency of nonlinear optical effects [9].

Time Dependence of the Output Intensity

The output intensity of a multimode laser can be written as

$$I(t) = \sum_{n,m} a_n a_m \exp\{i(n-m)\Omega t + \varphi_n - \varphi_m\} \qquad (1)$$

where Ω is the constant frequency spacing of adjacent axial modes, a_n the real amplitude, and φ_n the phase of the nth mode. If all the phases are equal ($\varphi_n = \varphi$), the output intensity consists of bandwidth-limited short pulses separated in time by the period $T = 2\pi/\Omega$. Such a pulse is shown in Fig. 1.

The influence of phase deviations, namely phases distributed statistically around the ideal value φ, was investigated in great detail [8], [9]. Such deviations result in an output consisting also of ideally short pulses but now superimposed on a fluctuating background that

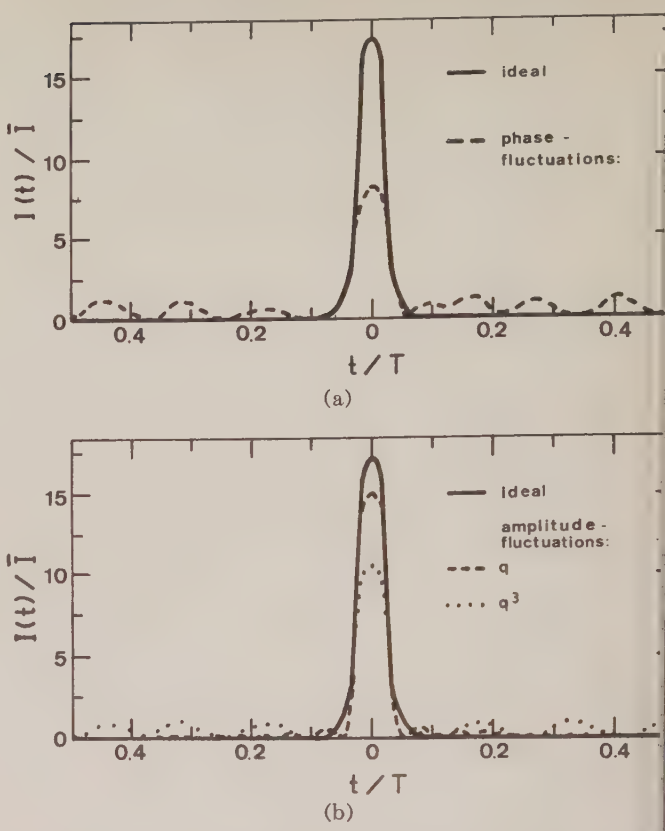


Fig. 1. Numerically calculated output intensity $I(t)$ for 31 mod having a Gaussian envelope. $\bar{I}$ is the mean intensity and T $2\pi/\Omega$ the pulse repetition time. (a) Phases φ_n randomly chos within the interval $(-\pi/2, +\pi/2)$, compared with ideal mo locking. (b) Fluctuating amplitudes a_n, generated by weighti with q and q^3, respectively, where q is randomly chosen with the interval $0 \leq q \leq 2$, compared with ideal mode locking.

becomes larger with increasing deviations. A numeric example, with the phases randomly distributed in t interval $-\pi/2 \leq \varphi_n \leq +\pi/2$, is given in Fig. 1(a). T calculation is based on 31 modes, where the value of t amplitudes is determined by a Gaussian envelope wi a standard deviation of eight modes.

A similar computation can be done for the alternati case that all the phases are equal and the amplitud a_n are chosen to have random deviations from the valu they had in the previous example. In Fig. 1(b) t examples of the instantaneous intensity are shown. the first one each amplitude of the ideal case was mul plied by a number q, chosen randomly within the interv $0 \leq q \leq 2$. In the second one, to increase the fluctuatio the ideal amplitude values were multiplied by q^3, whe q is again this random number. From Fig. 1(b) it is se that also for these resulting amplitude fluctuations t duration of the short pulse remains the same and fluctuating background appears.

Ensemble Averages

A somewhat more quantitative idea of the abo sketched situation is obtained by calculating the ensemb averaged values of the time-dependent intensity, ensem averaged with respect to phase variations ($\langle I \rangle_\varphi$) amplitude variations ($\langle I \rangle_a$). If an ensemble E is ch

rized by means of a normalized distribution function of the statistical variable x within E, the ensemble rage of a function $f(x, t)$ with respect to x is given by

$$\langle f\rangle_E(t) = \int_{-\infty}^{+\infty} f(x, t)p(x)\,dx. \tag{2}$$

plied to the intensity function of (1) this gives, for phase angles φ_n randomly distributed in the interval $\Phi, +\Phi$), the ensemble-averaged intensity

$$\langle I\rangle_\varphi(t) = \frac{1}{(2\Phi)^2}\int_{-\Phi}^{+\Phi} d\varphi_n \int_{-\Phi}^{+\Phi} d\varphi_m\, I(t, \varphi_n, \varphi_m). \tag{3}$$

shown earlier [8], [9] the explicit evaluation of this ression leads to:

$$\langle I\rangle_\varphi(t) = \bar{I}\left[\frac{\sin^2\Phi}{\Phi^2}\, i_p(t) + \left(1 - \frac{\sin^2\Phi}{\Phi^2}\right)\right] \tag{4}$$

ere $\bar{I}$ is the mean intensity (with respect to time) of multimode laser and $i_p(t) = I_p(t)/\bar{I}$ is the normalized, ally mode-locked pulse shape.

A similar ensemble average may be calculated with pect to amplitude variations, which are given by the tributions $p_n(a_n)$. Since the variations of the amplitudes assumed to be statistically independent, the ensemble-raged intensity becomes

$$(t)$$

$$\sum_{n\neq m} \exp\,[i(n - m)\Omega t] \int_0^\infty a_n p_n(a_n)\,da_n \int_0^\infty a_m p_m(a_m)\,da_m$$

$$+ \sum_n \int_0^\infty a_n^2 p_n(a_n)\,da_n$$

$$\sum_{n,m} \langle a_n\rangle\langle a_m\rangle \exp\,[i(n - m)\Omega t] + \sum_n (\langle a_n^2\rangle - \langle a_n\rangle^2), \tag{5}$$

ere (1) has been used and all phase angles have been equal to zero ($\varphi_n = 0$). Inspection of (5) shows that o in this case the ensemble-averaged emission consists a pulse of ideal shape $I_p(t)$ superimposed to a back-ound given by the sum of the variances of the fluctuating plitudes.

For the special case that all amplitudes a_n have the ne mean value $\langle a_n\rangle = a$ and are randomly distributed ound this value within the interval $(a - \Delta a, a + \Delta a)$, ensemble-averaged intensity of N_0 modes can be itten as

$$_a(t) = N_0 a^2[i_p(t) + \tfrac{1}{3}(\Delta a/a)^2] \qquad \Delta a \leq a, \tag{6}$$

ere $i_p(t) = I_p(t)/N_0 a^2$ is the normalized pulse of the ally mode-locked laser, and the condition $\Delta a \leq a$ rms that the amplitudes are always positive.

The difference between the effect of phase and amplitude variations is seen from (4) and (6). In both cases background intensity is introduced, but amplitude ctuations do not change the shape or intensity of the lse (i.e., the ideal pulse is retained) contrary to when ase fluctuations are present where the shape of the lse is conserved but its intensity is decreased. Particularly, even for maximum amplitude variations $\Delta a = a$ the pulse component does not vanish, whereas it completely disappears for maximum phase variations $\Phi = \pi$. The differences between phase and amplitude fluctuations will be discussed further on the basis of the intensity distribution.

Intensity Distribution and Its Moments

For fast intensity fluctuations, as in the output of wide-band solid-state lasers, only time integrated mean values of the intensity or of higher powers of the intensity can be measured. Therefore the emission is sufficiently described by the intensity distribution $p(I)$, which gives the probability to find the intensity I within the statistical ensemble and within the time of integration, i.e., a period of the emission [9]. The mean intensity of nonlinear optical effects, such as second or higher harmonic generation, is proportional to the corresponding higher order moment

$$\langle I^k\rangle = \int_0^\infty I^k p(I)\,dI \tag{7}$$

of the distribution $p(I)$. Their relative efficiency is $g^{(k)} = \langle I^k\rangle/\langle I\rangle^k$.

In the following, it turns out to be appropriate to first calculate for fixed t the distribution $P(\alpha, t)$ of the complex amplitude $\alpha(t)$ of the analytic signal

$$V(t) = \alpha(t)\exp\,(i\omega_0 t)$$
$$= \sum_{n=-N}^{+N} a_n \exp\,[i(n\Omega t + \varphi_n)]\exp\,[i\omega_0 t]. \tag{8}$$

This distribution contains even more information than $p(I)$ and $p(I)$ can be formally obtained by setting $I = |\alpha|^2$ and integrating in time. However, it will in general be more practical to calculate the time-dependent moments of the intensity

$$\langle I^k\rangle_E(t) = \int |\alpha|^{2k} P(\alpha, t)\,d^2\alpha, \tag{9}$$

averaging only with respect to the ensemble E, and to perform the time integration afterwards.

Amplitude Distribution for Amplitude Fluctuations

In this section, statistical deviations from an ideally mode-locked pulse with $\varphi_n = 0$ and equal amplitude $a_n = a$ of all modes in (8) are investigated. The deviations are considered to be caused by random and independent fluctuations of the individual amplitudes a_n. For mathematical convenience the amplitude fluctuations are assumed to be given by the same normalized distribution $p_a(a_n)$ for all modes, but still independent of each other. These assumptions are physically reasonable and exclude systematic deviations of the amplitudes, which correspond to mode selection not discussed in this context.

According to (8) the resulting complex amplitude $\alpha(t)$ is a sum of independently distributed amplitudes $a_n \exp\,\{in\Omega t\}$. Therefore the characteristic function $\chi_a(\lambda, t)$ of the amplitude distribution $P_a(\alpha, t)$ is given

by the product of the characteristic functions $\chi_n(\lambda, t)$ of the complex amplitudes of the individual modes [13]. If the complex numbers are represented by two-dimensional vectors, the characteristic function is given by

$$\chi_a(\lambda, t) = \int \exp[i\lambda\cdot\alpha] P_a(\alpha, t)\, d^2\alpha = \prod_n \chi_n(\lambda, t), \quad (10)$$

which is the Fourier transform of $P_a(\alpha, t)$. The characteristic functions $\chi_n(\lambda, t)$ are calculated using polar coordinates with $\lambda_x = \lambda \cos\vartheta$ and $\lambda_y = \lambda \sin\vartheta$:

$$\chi_n(\lambda, t) = \int_0^\infty \exp[i\lambda a_n \cos(\vartheta - n\Omega t)] p_a(a_n)\, da_n. \quad (11)$$

The exponential with the mean value $\langle a_n\rangle$ is split off,

$$\chi_n(\lambda, t) = \exp\{i\lambda\langle a_n\rangle \cos(\vartheta - n\Omega t)\}$$
$$\cdot \int_0^\infty \exp\{i\lambda(a_n - \langle a_n\rangle)\cos(\vartheta - n\Omega t)\} p_a(a_n)\, da_n, \quad (12)$$

and the remaining integrand is expanded into a power series with respect to λ and integrated term by term

$$\chi_n(\lambda, t) = \exp\{i\lambda\langle a_n\rangle \cos(\vartheta - n\Omega t)\}$$
$$\cdot[1 - \tfrac{1}{2}\lambda^2 \cos^2(\vartheta - n\Omega t)\langle(a_n - \langle a_n\rangle)^2\rangle + \cdots]. \quad (13)$$

Neglecting third and higher order terms in λ in the expansion, and approximating the resulting expression by an exponential, the product of the characteristic functions reads

$$\chi_a(\lambda, t) \cong \exp\{i\lambda\langle a\rangle \sum_n \cos(\vartheta - n\Omega t)$$
$$- \tfrac{1}{2}\lambda^2\sigma_a^2 \sum_n \cos^2(\vartheta - n\Omega t)\}, \quad (14)$$

where $\langle a\rangle$ is the mean value and $\sigma_a^2 = \langle a^2\rangle - \langle a\rangle^2$ is the variance of the amplitude distributions, which are the same for all modes. The truncation made to obtain (14) means that $P_a(\alpha, t)$ is approximated by a normal distribution. This is a good approximation for a large number of modes $N_0 = 2N + 1$ and physically reasonable amplitude distributions $p_a(a)$ satisfying the central limit theorem of statistics ([13], p. 193).

The final distribution $P_a(\alpha, t)$ is obtained by the inverse Fourier transformation, using Cartesian coordinates so that $\lambda_x = \lambda\cos\vartheta$, $\lambda_y = \lambda\sin\vartheta$ and $\alpha = (\alpha_x, \alpha_y)$.

$$P_a(\alpha, t) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} \exp[-i\lambda\cdot\alpha]\chi_a(\lambda, t)\, d\lambda_x\, d\lambda_y$$
$$= \frac{1}{2\pi B_a C_a} \exp\left\{-\frac{(\alpha_x - A_a)^2}{2B_a^2} - \frac{\alpha_y^2}{2C_a^2}\right\} \quad (15)$$

$$A_a = \langle a\rangle \sum_{n=-N}^{+N} \cos n\Omega t; \qquad B_a^2 = \sigma_a^2 \sum_{n=-N}^{+N} \cos^2 n\Omega t;$$

$$C_a^2 = \sigma_a^2 \sum_{n=-N}^{+N} \sin^2 n\Omega t.$$

The components of the resulting complex amplitude $\alpha(t) = \alpha_x + i\alpha_y$ of the analytic signal $V(t)$ are statistically independent and normally distributed with the time-dependent variances B_a^2 and C_a^2, respectively. The distribution of α_x is centered around the time-dependent mean value A_a, whereas α_y is always centered around zero. The random vector α may be represented as the sum of a constant vector and a so called Hoyt vector [14]. The x and y components of the latter are normally distributed with mean values zero, but with, in general, different variances.

Amplitude Distribution for Phase Fluctuations

The distribution of the resulting amplitude for phase fluctuations is calculated in the same manner as for amplitude fluctuations. In (8) the amplitudes are not constant and equal ($a_n = a$), whereas the phases φ_n are randomly and independently fluctuating. For mathematical convenience it is again assumed that the normalized distributions $p_\varphi(\varphi_n)$ are the same for all modes and furthermore that they are symmetrical around zero, i.e., $p_\varphi(\varphi_n) = p_\varphi(-\varphi_n)$. These assumptions are physically reasonable and avoid the possibility of additional systematic phase deviations, which have been studied elsewhere [9], [10].

The characteristic functions $\chi_n'(\lambda, t)$ are again calculated using polar coordinates for the Fourier transformation as in (11), but now integrating over the phases φ_n:

$$\chi_n'(\lambda, t) = \int_0^{2\pi} \exp\{i\lambda a \cos(\vartheta - n\Omega t - \varphi_n)\} p_\varphi(\varphi_n)\, d\varphi_n. \quad (16)$$

Separation of the exponential with the mean value $\langle\cos(\vartheta - n\Omega t - \varphi_n)\rangle$ and integration of the power series of the remaining expression term by term yields

$$\chi_n'(\lambda, t) = \exp\{i\lambda a \langle\cos(\vartheta - n\Omega t - \varphi_n)\rangle\}$$
$$\cdot[1 - \tfrac{1}{2}\lambda^2 a^2 s^2 + \cdots] \quad (17)$$

$$s^2 = \langle\cos^2(\vartheta - n\Omega t - \varphi_n)\rangle - \langle\cos(\vartheta - n\Omega t - \varphi_n)\rangle^2.$$

The characteristic function $\chi_\varphi(\alpha, t)$ of the distribution $P_\varphi(\alpha, t)$ is obtained by again replacing the expansion in (17) by an exponential, by using the generalized convolution theorem for independent statistical variables stated in (10), and by separating the fixed part ($n\Omega t$ — and the statistical part (φ_n) in the argument of the cosine functions. The result is

$$\chi_\varphi(\lambda, t) = \exp\left\{i\lambda a \langle\cos\varphi\rangle \sum_{n=-N}^{+N} \cos(n\Omega t - \vartheta) - \tfrac{1}{2}\lambda^2 a^2\right.$$
$$\left.\cdot\left[\sigma_c^2 \sum_{n=-N}^{+N} \cos^2(n\Omega t - \vartheta) + \sigma_s^2 \sum_{n=-N}^{+N} \sin^2(n\Omega t - \vartheta)\right]\right\} \quad (18)$$

$$\sigma_c^2 = \langle\cos^2\varphi\rangle - \langle\cos\varphi\rangle^2;$$

$$\sigma_s^2 = \langle\sin^2\varphi\rangle - \langle\sin\varphi\rangle^2 = \langle\sin^2\varphi\rangle.$$

In this equation the variances σ_c^2 of $\cos\varphi$ and σ_s^2 of $\sin$ are introduced. They are mutually dependent since $\sigma_c^2 + \sigma_s^2 = 1 - \langle\cos\varphi\rangle^2$ and are a measure of the width of the phase fluctuations. Equation (18) is again a ge

roximation, according to the central limit theorem, a large number of modes and physically reasonable se distributions.

he inverse Fourier transformation, again using tesian coordinates so that $\lambda_x = \lambda \cos \vartheta$, $\lambda_y \sin \vartheta$, $\alpha = (\alpha_x, \alpha_y)$, leads to expressions that are quite ilar to those obtained in the case of amplitude fluctions, namely,

$$\alpha, t) = \frac{1}{2\pi B_\varphi C_\varphi} \exp\left\{-\frac{(\alpha_x - A_\varphi)^2}{2B_\varphi^2} - \frac{\alpha_y^2}{2C_\varphi^2}\right\} \tag{19}$$

$$A_\varphi = a \langle \cos \varphi \rangle \sum_{n=-N}^{+N} \cos n\Omega t$$

$$B_\varphi^2 = a^2\left[\sigma_c^2 \sum_{n=-N}^{+N} \cos^2 n\Omega t + \sigma_s^2 \sum_{n=-N}^{+N} \sin^2 n\Omega t\right]$$

$$C_\varphi^2 = a^2\left[\sigma_s^2 \sum_{n=-N}^{+N} \cos^2 n\Omega t + \sigma_c^2 \sum_{n=-N}^{+N} \sin^2 n\Omega t\right].$$

king the above approximations, the amplitude distrition resulting from phase fluctuations is again given normal distributions of the two independent comnents α_x and α_y, but with other equations for the center ue A_φ and the variances B_φ^2 and C_φ^2. In a different ntext [15] it is shown that the distribution of the sum ctor of a large number of random vectors with even re general distributions of phases and amplitudes ays leads, in the approximation of the central limit orem, to a distribution of the type of (15) and (19) a constant time t. The distributions $P(\alpha, t)$ will be cussed later.

Time-Dependent Moments of the Intensity Distribution

By means of $P(\alpha, t)$ given by (15) and (19), the insity distribution at time t can be calculated using relation $I = |\alpha|^2$, but the result is rather complicated]. The time-independent intensity distribution $p(I)$, ich specifies the intensity variations within the statical ensemble and within a period of the emission, even more difficult to obtain, because the necessary ne integration can not be performed analytically. wever, only the moments of $p(I)$ are of basic interest nonlinear optical experiments and they can be obtained alytically by first calculating the time-dependent ments $\langle I^k \rangle_E(t)$ with respect to the ensemble E and erwards integrating in time.

The time-dependent moments $\langle I^k \rangle_E(t)$ can be found th the relation $I^k = (\alpha_x^2 + \alpha_y^2)^k$. The necessary moments $^l \alpha_y^{2(k-l)}\rangle$ of the amplitude components are calculated her from the distribution $P(\alpha, t)$ by integration or m the characteristic function $\chi(\lambda, t)$ by differentiation.

$$^l \alpha_y^{2(k-l)}\rangle = \int_E \alpha_x^{2l} \alpha_y^{2(k-l)} P(\alpha, t)\, d^2\alpha = (i)^{-2k} \frac{\partial^{2k} \chi(\lambda, t)}{\partial \lambda_x^{2l}\, \partial \lambda_y^{2(k-l)}}\bigg|_{\lambda=0} \tag{20}$$

ne general form of the ensemble-averaged intensity according to (15) and (19), respectively, is

$$\langle I \rangle_E(t) = A^2 + B^2 + C^2. \tag{21}$$

The ensemble average of the square of the intensity reads

$$\langle I^2 \rangle_E(t) = A^4 + 3(B^2 + C^2)^2 + 2A^2(B^2 + C^2) + 4B^2(A^2 - C^2). \tag{22}$$

In the case of amplitude fluctuations A, B, and C are given by [15], so that $\langle I \rangle_a(t)$ reads explicitly:

$$\langle I \rangle_a(t) = \langle a \rangle^2 \sum_{n,m=-N}^{+N} \cos n\Omega t \cos m\Omega t + N_0 \sigma_a^2, \tag{23}$$

where $N_0 = 2N + 1$ is the total number of modes. This is the same result as obtained from (5) for the special case of equal-mean amplitudes and equal-amplitude distributions. The time average $(\langle\ \rangle_t)$ is performed by integrating (23) term by term over one period of Ω. The result is

$$\langle \langle I \rangle_a \rangle_t = \bar{I} = N_0[\langle a \rangle^2 + \sigma_a^2] = N_0 \langle a^2 \rangle, \tag{24}$$

which is the sum of the mean intensities $\langle a^2 \rangle$ of the individual modes. In the same way $\langle \langle I^2 \rangle_a \rangle_t$ is calculated to be

$$\langle \langle I^2 \rangle_a \rangle_t = \tfrac{2}{3} N_0^3 \langle a \rangle^4 + N_0^2(5\sigma_a^2 \langle a \rangle^2 + 2\sigma_a^4) + N_0(\tfrac{1}{3}\langle a \rangle^4 + \sigma_a^4) + \sigma_a^2 \langle a \rangle^2. \tag{25}$$

Inserting the expressions of (19) into (21) and (22), the corresponding equations for phase fluctuations are found to be

$$\langle I \rangle_\varphi(t) = N_0 a^2 + \langle \cos \varphi \rangle^2 a^2 \cdot \left[\sum_{n,m=-N}^{+N} \cos n\Omega t \cos m\Omega t - N_0\right] \tag{26}$$

$$\langle \langle I \rangle_\varphi \rangle_t = \bar{I} = N_0 a^2 \tag{27}$$

$$\langle \langle I^2 \rangle_\varphi \rangle_t = \{\tfrac{2}{3} N_0^3 \langle \cos \varphi \rangle^4 + N_0^2[2(1 - \langle \cos \varphi \rangle^4) + \langle \cos \varphi \rangle^2(\sigma_c^2 - \sigma_s^2)] + N_0[\tfrac{1}{3} \langle \cos \varphi \rangle^4 + (\sigma_c^2 - \sigma_s^2)^2] + (\sigma_c^2 - \sigma_s^2) \langle \cos \varphi \rangle^2\} a^4. \tag{28}$$

Equation (26) is again in agreement with the more general (4).

Discussion and Comparison of the Results

The time dependence of the parameters A, B, and C of the amplitude distributions for $N_0 = 17$ is illustrated in Fig. 2(a) for amplitude fluctuations and in Fig. 2(b) for phase fluctuations. The original pulseform is essentially retained in the center parameter A. The standard deviations B and C of the components α_x and α_y are significantly different during the pulse duration, but between the pulses they both have approximately the same value. This corresponds to a randomly fluctuating (Gaussian) background. It is interesting to note that the fluctuations of the transverse α_y component at the pulse maximum ($t = 0$) only vanish in the case of amplitude variations.

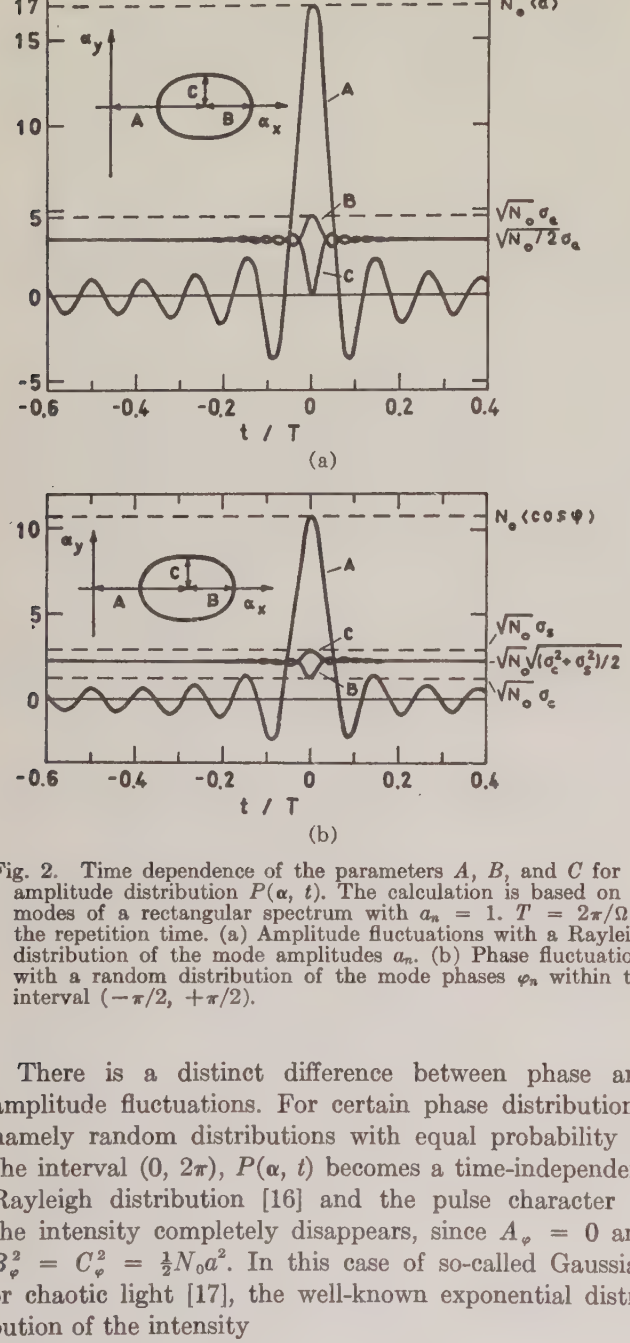


Fig. 2. Time dependence of the parameters A, B, and C for an amplitude distribution $P(\alpha, t)$. The calculation is based on 17 modes of a rectangular spectrum with $a_n = 1$. $T = 2\pi/\Omega$ is the repetition time. (a) Amplitude fluctuations with a Rayleigh distribution of the mode amplitudes a_n. (b) Phase fluctuations with a random distribution of the mode phases φ_n within the interval $(-\pi/2, +\pi/2)$.

There is a distinct difference between phase and amplitude fluctuations. For certain phase distributions, namely random distributions with equal probability in the interval $(0, 2\pi)$, $P(\alpha, t)$ becomes a time-independent Rayleigh distribution [16] and the pulse character of the intensity completely disappears, since $A_\varphi = 0$ and $B_\varphi^2 = C_\varphi^2 = \frac{1}{2}N_0a^2$. In this case of so-called Gaussian or chaotic light [17], the well-known exponential distribution of the intensity

$$p(I) = (N_0a^2)^{-1} \exp[-I/N_0a^2] = \bar{I}^{-1} \exp[-I/\bar{I}] \quad (29)$$

follows directly from (19) and is apparently independent of the time t. On the other hand, the distribution $P_a(\alpha, t)$ for amplitude fluctuations, (15), is always time dependent.[1] It thus follows that Gaussian or completely chaotic light can not be generated by fluctuating amplitudes of the modes. A phase-locking behavior will always be observed, although the fluctuating background between the pulses may vary according to the size of the fluctuations. Note that statistical-amplitude variations of the modes introduce background fluctuations of width essentially equal to the inverse bandwidth even fo mode-locked lasers with a frequency chirp [2] (systemati phase deviations [9], [10]). Such lasers normally emi pulses that are longer than the reciprocal bandwidth

Some basic parameters necessary to calculate th mean value A and the variances B^2 and C^2 shall now b stated for a few distributions of special physical interest In the case of amplitude fluctuations, the rectangula distribution around the mean amplitude value $\langle a \rangle$ wit a total width $2\Delta a$ and the Rayleigh distribution $(a/\gamma^2$ $\exp[-a^2/2\gamma^2]$, which corresponds to an exponential o chaotic intensity distribution of the individual modes yield

rectangular:

$$\sigma_a^2 = \tfrac{1}{3}(\Delta a)^2 \leq \tfrac{1}{3}\langle a \rangle^2 \quad (30a$$

Rayleigh:

$$\sigma_a^2 = \frac{4}{\pi}\langle a \rangle^2, \qquad \gamma^2 = \frac{2}{\pi}\langle a \rangle^2. \quad (30b$$

Obviously, the amplitude distribution function $p_a(a_n)$ i always equal to zero for negative arguments. For phas fluctuations, the distribution must be symmetric aroun zero. The rectangular distribution in the interva $(-\Phi, +\Phi)$ and the normal distribution $(\sqrt{2\pi}\ \sigma)^-$ $\exp\{-\varphi^2/2\sigma^2\}$ yield

rectangular:

$$\langle \cos\varphi \rangle = \frac{\sin\Phi}{\Phi} = \operatorname{sinc}\Phi; \quad (\sigma_c^2 - \sigma_s^2) = \operatorname{sinc} 2\Phi - \operatorname{sinc}^2 \Phi \quad (31a$$

normal:

$$\langle \cos\varphi \rangle = \exp\{-\sigma^2/2\}; \quad (\sigma_c^2 - \sigma_s^2) = e^{-\sigma^2}(e^{-\sigma^2} - 1). \quad (31b$$

The average relative efficiency of second-order nonlinea optical effects, such as second-harmonic generation o two photon absorption, for a fluctuating fundamenta intensity with respect to constant intensity, is given by [9] $g^{(2)} = \langle\langle I^2\rangle_E\rangle_t/\bar{I}^2$. For amplitude fluctuations $g^{(2)}$ is found from (24) and (25) and reads in the approximation for $N_0 \gg 1$

$$g^{(2)} \cong [\tfrac{2}{3}N_0 + (5\beta + 2\beta^2)]/(1 + \beta)^2. \quad (32$$

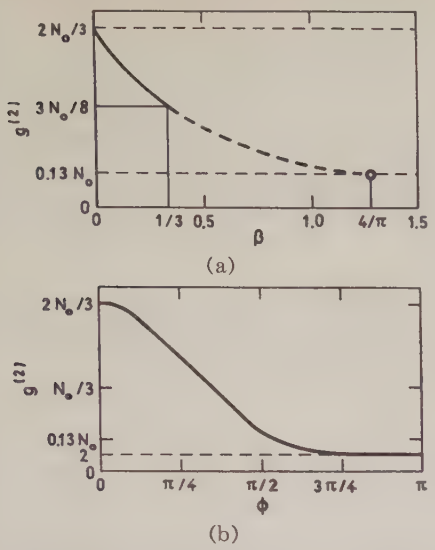
The relative efficiency $g^{(2)}$ depends only on the ratio $\beta = \sigma_a^2/\langle a \rangle^2$ of the amplitude distribution. The dependence of $g^{(2)}$ on β is plotted in Fig. 3(a). The rectangular distribution with $\beta = \frac{1}{3}(\Delta a/\langle a \rangle)^2 \leq \frac{1}{3}$ is represented by the solid line and the Rayleigh distribution by the point $\beta = 4/\pi$.

For phase fluctuations $g^{(2)}$ is found from (27), (28) and (31) and reads, again in the approximation for $N_0 \gg 1$

rectangular:

$$g^{(2)} \cong \tfrac{2}{3}N_0 \operatorname{sinc}^4\Phi + (2 - 3\operatorname{sinc}^4\Phi + \operatorname{sinc} 2\Phi \operatorname{sinc}^2\Phi), \quad (33a$$

[1] Note that the variance σ_a^2 is limited by the finite mean intensity $\bar{I}$, (24), and that even fluctuations allowing negative amplitudes (or $\varphi_n = 0, \pi$) are not sufficient to generate chaotic light.



3. Average relative efficiency $g^{(2)} = \langle\langle I^2\rangle_E\rangle_t/\bar{I}^2$ for second-der nonlinear optical effects ($N_0 \gg 1$). (a) Amplitude fluctua-ns with a normalized variance $\beta = \sigma_a^2/\langle a\rangle^2$. The solid line presents the rectangular distribution and the point at $\alpha = 4/\pi$ e Rayleigh distribution of the mode amplitudes. (b) Random ctuations of the phases φ_n in the interval $(-\Phi, +\Phi)$. The niting value $g^{(2)} = 2$ is reached for $\Phi = \pi$. The curve is plotted r $N_0 = 30$.

nal:

$$g^{(2)} \cong \tfrac{2}{3}N_0 e^{-2\sigma^2} + (2 - 3e^{-2\sigma^2} + e^{-3\sigma^2}). \qquad (33b)$$

ation (33a) is plotted in Fig. 3(b).

comparison of Fig. 3(a) and (b) shows that both es assume the value $g^{(2)} = \frac{2}{3}N_0$ for zero fluctuations $= 0$, $\Phi = 0$), in agreement with earlier direct cal-tions [18]. For amplitude fluctuations, the relative iency drops only to a minimum value of $g^{(2)} = \frac{1}{3}N_0$, namely for a Rayleigh distribution. In contrast his, the minimum value for the phase fluctuations is $= 2$, namely for $\Phi = \pi$, and is independent of the nber of modes N_0, in agreement with the well-known lt for Gaussian light or free-running lasers [9], [19].

Conclusions

or both statistical amplitude and phase variations he modes of a mode-locked laser, the Cartesian com-ents of the complex amplitude of the resulting output to a good approximation normally distributed. The ribution is in general time dependent. As long as systematic amplitude or phase variations from the lly mode-locked pulse are introduced, the pulse pe remains essentially constant but appears super-osed on a fluctuating background. Even for maximum olitude variations, given by a Rayleigh distribution, pulsed output behavior is maintained. But for phase ations, the pulse behavior disappears completely he fluctuating background for a random phase distri-ion over 2π.

similar difference is also observed in the change he relative efficiency of second-order nonlinear pro-es with respect to constant fundamental intensity. recent experiment [6] it was shown, by using third-monic generation, that in fact a larger amount of energy (in their case 96 percent) is contained in a background between the pulses. For ideally pulsed output the efficiency is proportional to the number of the locked modes. For maximum amplitude variations (Rayleigh distribution), the efficiency is reduced to about 20 percent of the initial ideally mode-locked value, whereas for maximum phase variations, randomly distributed over 2π, the efficiency becomes, independently of the number of modes, twice as large as for constant intensity, which is a well-known result for Gaussian light or a free-running laser.

Acknowledgment

The authors wish to thank Prof. K. P. Meyer for giving the opportunity to perform this work and for his continuous encouragement, and are indebted to Dr. K. Weiss for helpful discussions and suggestions.

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OPTO-ELECTRONICS 2 (1970) 63-67

The Influence of Spatial Hole-Burning in Single Mode Q-Switch Operation

H. WEBER, A. A. GRUTTER
Institute of Applied Physics, University of Berne, Switzerland

Received 17 March 1970

The influence of spatial hole-burning effects on the output of Q-switched giant pulse lasers is investigated. Generalised fundamental equations are established and analytically treated to obtain the total output energy and the remaining inversion density. The results are completed by numerical computations which made it possible to obtain the time-dependent output intensity and the width of the pulses. The solutions of the modified equations are compared with earlier calculations where spatial hole-burning has not been included. It is shown that the formation of standing waves inside the cavity leads to a reduction of the total output energy but does not affect the pulse width of the laser emission.

1. Introduction

In several papers [1–3] the fundamental equations that describe the mechanism of single mode Q-switching have been established and implicitly solved for cases where any spatial and frequency hole-burning effects have been neglected. This means that the inversion density is destroyed homogeneously as the laser pulse builds up. The output power and the total output energy have been calculated. A theoretical pulse duration has been defined [4] and calculated explicitly as a function of the resonator length, loss, gain and reflectivity [5]. Experimental comparisons usually show a considerable departure from these ideal results. The measured pulses have lower peak powers and longer widths than those theoretically predicted. There are several explanations for these discrepancies between theory and experiment. For a non-uniform pump light density with a resulting maximum inversion in the rod centre the giant pulse will start in low-order modes and spread into higher-order transverse modes during its evolution [10, 11]. Non-uniform pumping thus tends to spread the giant pulse in time. Other effects determining the exact shape of the output pulse include travelling wave phenomena of the photon density inside the cavity [12], and spatial hole-burning as a result of standing waves inside the cavity, which is discussed in this paper. Earlier investigations on this subject [13] are enlarged by calculating the explicit expressions of the time-dependence of the inversion density, photon density, and the output intensity of the laser, starting from modified fundamental equations. The effect of spatial hole-burning on the pulse width and pulse height is discussed and compared to the ideal case without hole-burning.

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2. Fundamental Equations and Analytical Results

For one mode of longitudinal order m inside a homogeneous cavity of length L the photon density $q(t)$ is given by the slowly-varying envelope approximation, which may be used when the density does not vary considerably during a round trip time $2L/c$:

$$q(t, z) = q_0(t) \sin^2 m\pi z = \tfrac{1}{2}(1 - \cos 2m\pi z)\, q_0(t). \quad (1)$$

$z = x/L$ is the normalised space co-ordinate in the direction of the resonator axis. This formulation of the photon density is valid for a plane parallel resonator and neglects transversal mode structure. The fundamental equations describing Q-switching without hole-burning are thus modified to give [6]

$$\frac{\mathrm{d}\Delta n}{\mathrm{d}t} = -\Delta n B q_0 (1 - \delta \cos 2\pi m z) \quad (2a)$$

$$\frac{dq_0}{dt} = q_0 \left[B \int_0^1 \Delta n (1 - \delta \cos 2\pi mz)\, dz - 1/t_R \right], \quad \ldots (2b)$$

with the initial condition

$$\Delta n\,(0, z) = \Delta n_a \qquad (0 \leqslant z \leqslant 1) \qquad (3)$$

where $\Delta n = n_2 - n_1$ is the inversion density, B the Einstein coefficient and t_R the resonator time constant, which may be expressed by means of the mean reflectivity $R = \sqrt{R_1.R_2}$ of the two cavity mirrors, the loss factor V and cavity length L in the form $t_R = -L/c\, ln(RV)$. c is the velocity of light. The parameter δ assumes the value 0 for the case without hole-burning and 1 for complete hole-burning. For further treatment it is convenient to write equations 2 in the following normalised form:

$$\frac{dx}{ds} = -xy\,[1 - \delta \cos 2\pi\alpha] \qquad (4a)$$

$$\frac{dy}{ds} = y \left[\int_0^1 x(1 - \delta \cos 2\pi\alpha)\, d\alpha - x_{th} \right] \qquad (4b)$$

thereby using the normalised inversion density $x = \Delta n/\Delta n_a$, threshold inversion

$x_{th} = \Delta n_{th}/\Delta_{na} = 1/Bt_R\Delta n_a$,

photon density $y = q_0/\Delta n_a$, space co-ordinate $\alpha = mz$ and the time coordinate $s = \Delta n_a Bt$.

Equation 4 gives the fundamental relations which describe the mechanism of Q-switching when one mode is present in the cavity and the effect of spatial hole-burning is considered. These equations look quite complicated and they are not easy to integrate analytically. They were therefore treated with computer methods which will be discussed in the next section. Nevertheless, it is possible to obtain some significant physical results without tedious numerical calculations.

As is shown in the appendix, equations 4 may be analytically integrated to give a relation between the instantaneous normalised photon density y and the normalised energy

$$\epsilon(s) = \int_0^s y(\sigma)\, d\sigma$$

for a given initial photon density y_a*:

$$y - y_a = 1 - e^{-\epsilon} I_0(\delta\epsilon) - \epsilon x_{th} \qquad (5)$$

In this expression I_0 stands for a modified Bessel function of zeroth order [9]. At time $s = \infty$ the photon density will again have reached its initial value y_a and the corresponding energy parameter $\epsilon_\infty \equiv \epsilon(\infty)$ will then represent the total energy emitted during the Q-switch operation. Equation 5 then becomes:

$$x_{th} = [1 - e^{-\epsilon_\infty} I_0(\delta\epsilon_\infty)]/\epsilon_\infty \,. \qquad (6)$$

This function is pictured in fig. 1, which shows that the total output energy for the case of ideal

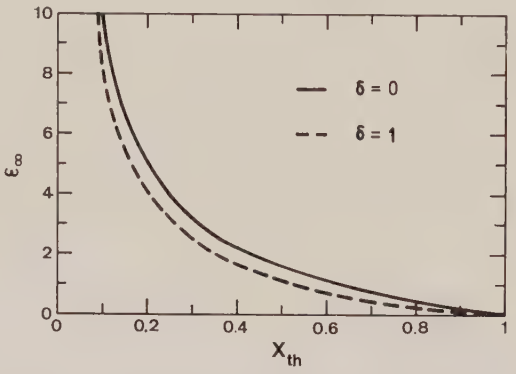

Figure 1 The dependence of the total normalised output energy ϵ_∞ of a laser pulse on the inversion parameter x_{th} for the cases where spatial hole-burning effects are considered ($\delta = 1$) or neglected ($\delta = 0$).

hole-burning ($\delta = 1$) is always smaller than without hole-burning ($\delta = 0$), for a given threshold inversion parameter x_{th}. This result is readily understandable since hole-burning effects prevent an optimal reduction of the inversion density. But as the difference between the two cases is rather small, it is clear that additional axial modes would not bring a considerable increase of the total output energy, although the inversion reduction process would proceed more homogeneously.

The maximum output power y_m may also be obtained by analytical calculation. As is shown in the appendix the maximum occurs when the energy parameter ϵ_m fulfils the condition:

$$x_{th} = e^{-\epsilon_m}[I_0(\delta\epsilon_m) - I_1(\delta\epsilon_m)] \,, \qquad (7)$$

which for no spatial hole-burning ($\delta = 0$) reduces to

$$x_{th} = e^{-\epsilon_m} \,. \qquad (8)$$

The corresponding output power then follows from equation 5:

$$y_m = y_a + 1 - e^{-\epsilon_m} I_0(\delta\epsilon_m) - \epsilon_m x_{th} \,, \qquad (9)$$

and equation 7, by eliminating the energy para-

*The corresponding energy E and power P in watt sec and watt respectively are obtained from the relations $E = (\frac{1}{2}\, hvTFc/B).\epsilon$ and $P = (\frac{1}{2}\, hvcFT\Delta n_a).y$; F = laser beam cross section, T = mirror transmission.

meter ϵ_m. For the case ($\delta = 0$) equations 9 and 7 reduce to

$$y_m = y_a + 1 - x_{th} + x_{th} ln x_{th} \ . \qquad (10)$$

This result has been previously obtained [4] and appears here as a special case of more general calculations. It is illustrated, together with the corresponding result for spatial hole-burning ($\delta = 1$), in fig. 2. A comparison with fig. 1 shows

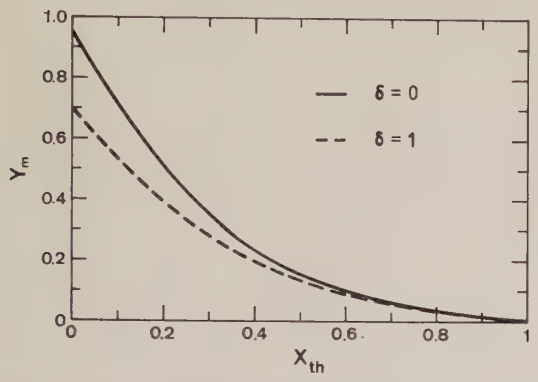

Figure 2 The normalised photon density y_m of the maximum pulse height as a function of the inversion parameter x_{th} with ($\delta = 1$) and without ($\delta = 0$) spatial hole-burning.

that the maximum output power also becomes smaller for every inversion parameter x_{th} as soon as the inversion density is inhomogeneously destructed.

3. Time-Dependent Solutions and Pulse Durations

The fundamental equations 5 have been numerically integrated by means of an electronic computer with Runge-Kutta methods to obtain the time-dependent photon density $y(s)$ and the time- and space-dependent inversion density $x(\alpha, s)$ for various inversion factors x_{th}. One typical example of the output intensity for an inversion factor $x_{th} = 0.6$ is shown in fig. 3. It is seen from the figure that the effect of hole-burning reduces the pulse height effectively in accordance with the analytical results of section 2. Furthermore, it is seen that the pulse width at half height is practically the same with or without hole-burning. For several inversion parameters this numerically calculated pulse width is shown, in fig. 4, to tend to infinity for both maximum and negligible losses, in accordance with earlier results [4]. The minimum value depends on both the loss and gain factors of the laser, so that two pulses of equal pulse width may differ in energy.

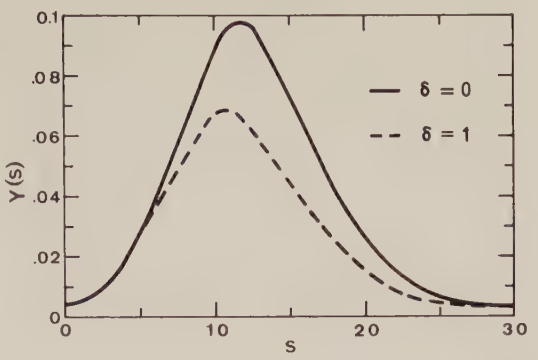

Figure 3 The time-dependent normalised photon density $y(s)$ of a laser pulse calculated by numerically integrating the fundamental Q-switch equations for an inversion factor $x_{th} = 0.6$. The solid line is valid when no hole-burning effects are considered ($\delta = 0$).

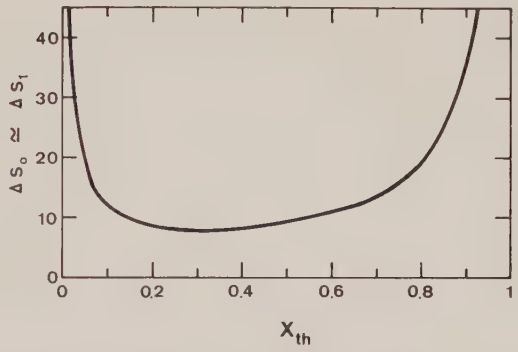

Figure 4 Numerically calculated pulse width Δs for different inversion parameters x_{th}. Hole-burning does not effectively change the ideal pulse width, so that $\Delta s_0(\delta = 0)$ is approximately equal to $\Delta s_1(\delta = 1)$.

For a direct comparison with experimental results it has to be noted that the conditions for minimum pulse width are not identical with those for maximum output power or output energy [2].

During the build-up of the pulse the inversion density $x(s)$ is destroyed. After the pulse has been emitted some inversion density is left over. This situation is pictured in fig. 5. At time $s = \infty$ there remains an inversion density inside the cavity which is formed according to the shape of the standing waves of the oscillating mode. The smaller the losses, the more the photon density inside the cavity is able to destroy large parts of the inversion density. There always remains, however, a train of smaller or larger antinodes (one example of these is pictured in fig. 5) indicating the places where the nodes of the standing waves, formed by the laser mode, have been.

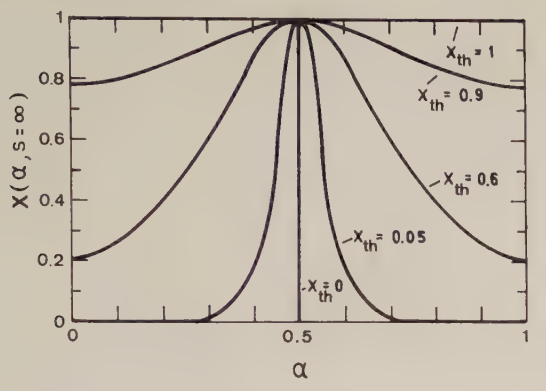

Figure 5 The normalised inversion density inside the cavity that remains at time $s = \infty$, when the pulse has been emitted, against distance parameter, α. Spatial hole-burning produces a modulated inversion density, which characteristically depends on the loss parameter x_{th}.

In an actual experiment the effect of spatial hole-burning may be considerably reduced when several modes are oscillating. Then the antinodes formed by the modes of different orders are generally located at different places and overlap each other, which again gives a more or less even destruction of the inversion density in the cavity. Lattice interactions of the pumped levels with neighbouring unexcited atoms may also smooth out the spatial inhomogenities of the inversion density. However, for ruby lasers this sort of diffusion occurs with a diffusion time of 5×10^{-4} sec[7] which is much larger than the duration of the Q-switch pulses (~nsec), and has therefore not been considered here.

4. Conclusions

The present calculations have shown that spatial hole-burning reduces the total output energy and maximum output power, since the standing wave inside the cavity cannot destroy the inversion density completely. There remains some inversion density in the cavity, which could eventually be destroyed by higher-order axial modes. However the difference between ideal hole-burning and no hole-burning is small, so that the output energy and output power would not increase considerably in the case of axial multimode operation. The situation is quite different however, if transversal modes are taken into account. This case is not discussed here.

Although the modified equations give a reduced total output energy, the pulse width is not changed within numerical accuracy. Any experimentally observed deviations from the ideal pulse width as obtained from the usual Q-switch equations may not therefore, be attributed to spatial hole-burning effects.

Appendix

Implicit Solutions of the Fundamental Equations

To obtain an implicit solution of the two coupled differential equations [4], the time parameter s is first eliminated by dividing the two equations by each other:

$$\frac{dy}{dx} = -\left\{ x(1 - \delta \cos 2\pi\alpha) \right\}^{-1} \int_0^1 x\,(-\delta \cos 2\pi\alpha)\, d\alpha - x_{th} \Big\} . \qquad \text{(A1)}$$

An integration of this equation would give the instantaneous photon density y as a function of the instantaneous inversion density x. Experimentally it is however more interesting to know the dependence of y on the total output energy of the laser emission at time s

$$\epsilon = \int_0^s y(\sigma)\, d\sigma ,$$

which is related to x by:

$$x(\alpha, s) = e^{-\epsilon(1-\delta \cos 2\pi\alpha)} \qquad \text{(A2)}$$

as is immediately seen by formally integrating equation 4a. The differential equation for y as a function of x can therefore be transformed to an equation for y as a function of ϵ:

$$\frac{dy}{d\epsilon} = \int_0^1 e^{-\epsilon(1-\delta \cos 2\pi\alpha)}(1 - \delta \cos 2\pi\alpha)\, d\alpha + x_{th} . \qquad \text{(A3)}$$

An integration with the initial conditions $y = y_a$ and $\epsilon = 0$, which are valid at time $s = 0$ then leads to the solution:

$$y - y_a = 1 - e^{-\epsilon} \int_0^1 e^{\epsilon\delta \cos 2\pi\alpha} d\alpha + \epsilon x_{th} . \qquad \text{(A4)}$$

The remaining definite integral is known as a modified Bessel function of zeroth order [9] so that equation A5 finally takes the form:

$$y - y_a = 1 - e^{-\epsilon} I_0(\delta\epsilon) - \epsilon x_{th} . \qquad \text{(A5)}$$

This function assumes its maximum value when the first derivative with respect to ϵ is equal to zero:

$$\frac{dy}{d\epsilon} = e^{-\epsilon} I_0(\delta\epsilon) - e^{-\epsilon}\delta I_1(\delta\epsilon) - x_{th} = 0 . \qquad \text{(A6)}$$

In this expression use has been made of the

relation that the first derivative of the modified Bessel function of zeroth order is equal to the modified Bessel function of first order I_1. [9] From equation A6 the energy parameter ϵ_m, for which the output intensity is a maximum, can then be determined. For the case of no spatial hole-burning ($\delta = 0$) it becomes:

$$\epsilon_m = -lnx_{th} \ . \qquad \text{(A7)}$$

The corresponding output intensity follows from equation (A5) and is further discussed in section 2.

Acknowledgement

The authors wish to thank Professor K. P. Meyer for giving them the opportunity to perform this work and for his continuous encouragement.

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